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# Differentiability of Continuous Convex Functions

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Mathematical Analysis

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# Diferencovatelnost konvexní spojité funkce

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# Contents

# 1 Introduction

This thesis is about the differentiability of continuous convex functions on topological vector spaces. The mentioned chapters and sections are the section of the thesis. First we introduce the basic definitions of different types of differentiability (chapter 1), then we describe the relationships between these differentiabilities and give the corresponding diagram (section 2.1 and 2.2), followed by the special case of a scalar function and the corresponding diagram (section 2.3). In the section 2.4 we describe the relationship between differentability and continuity.

The goal of this thesis is to classify the relationships between various types of differentiabilities for the case of continuous convex functions and provide the corresponding diagram and the counter-examples showing that implications on the diagram are not equivalence relations (section 2.5).

# 2 Main results

## 2.1 A construction of convex functions

We give a construction of convex functions on infinite-dimensional spaces and then apply it to give an illustration to a theorem from [6] on existence of Gâteaux differentiability points that are not Fréchet differentiability points (for more detailed information, see [8]), viz., we construct on $l_p$, $p \geq 1$, a convex continuous function, which is everywhere compactly differentiable and is not Fréchet differentiable at zero.

It is known that, in normed spaces, Fréchet differentiability at a point implies compact differentiability at this point. On the other hand, there are compactly differentiable functions on a normed space which are not Fréchet differentiable (see [1], [3]).

For continuous convex functions on normed spaces, Gâteaux differentiability at a point implies compact differentiability at this point (see [4], [6]). More than that, for functions on arbitrary topological vector spaces, Gâteaux differentiability of a continuous convex function at a point implies Michal-Bastiani differentiability at this point [4]. Michal-Bastiani differentiability (see [3]) is in general stronger than compact differentiability, and for normed spaces is equivalent to compact differentiability.

**Theorem.** *Let $X$ be a set, let $F(X,\mathbb{R})$ be a space of real-valued functions on $X$, and let $p$ be a convex function on $F(X,\mathbb{R})$. Let $\prec$ denote the following order relation on $F(X,\mathbb{R})$:*

$$f \prec g :\Leftrightarrow f(x) \leq g(x) \quad \forall x \in X,$$

*and assume that $p$ is monotonically increasing with respect to $\prec$ on nonnegative functions. Let, further, for each $x \in X$ there is given a nonnegative convex function $\varphi_x$ on $\mathbb{R}$, such that for each $f \in F(X,\mathbb{R})$ the function*

$$x \mapsto \varphi_x(f(x))$$

*also belongs to $F(X,\mathbb{R})$, and let $A$ be an operator*

$$A : F(X,\mathbb{R}) \to F(X,\mathbb{R})$$

*defined by the formula*

$$(Af)(x) = \varphi_x(f(x)).$$

*Then $p \circ A$ is a convex function on $F(X,\mathbb{R})$.*

Here we provide the mentioned application of the above theorem.

**Theorem.** *Define a function $f : l_p \to \mathbb{R}$, $p \geq 1$, by*

$$f(x) = ||(|x_1|^{1+1}, |x_2|^{1+\frac{1}{2}}, |x_3|^{1+\frac{1}{3}}, \ldots)||_p,$$

*where*

$$||x||_p := \sqrt[p]{\sum_i |x_i|^p}.$$

*We claim that this function is defined on the whole space $l_p$, is continuous and convex, is everywhere compactly differentiable, but is not Fréchet differentiable at zero.*

## 2.2 Continuous convex MS-differentiable function need not be HL-differentiable

We construct here an example of a continuous convex function on a locally convex space, which is MS-differentiable at a point, but is not HL-differentiable at this point.

There are three the most interesting differentiabilities: MB-differentiability, MS-differentiability (in the sense of Marinescu-Sebastião e Silva), and HL-differentiability (in the sense of Hyers-Lang), with the following relations between:

$$\mathrm{HL} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \mathrm{MS} \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} \mathrm{MB}$$

(see [3]). It is known long ago (see, e.g. [6]) that for continuous convex functions

$$\mathrm{MB} \not\Rightarrow \mathrm{MS}$$

(already for functions on normed spaces, for which MB-differentiability is reduced to compact differentiability and MS-differentiability to Fréchet differentiability). Here we show that for continuous convex functions

$$\mathrm{MS} \not\Rightarrow \mathrm{HL}.$$

We construct explicitly a continuous convex function on a non-normable locally convex space, which is MS-differentiable at a point, but is not HL-differentiable at this point. The non-normability is essential, since for normed spaces both MS- and HL-differentiabilities are equivalent to Fréchet differentiability [3].

First we introduce several needed lemmas:

**Lemma.** *1) A mapping $r : X \to \mathbb{R}$ with $r(0) = 0$ is* MS*-small if and only if*

$$\exists U \in Nb_0(X)\ \forall \varepsilon > 0\ \exists U' \in Nb_0(X)\ :\ (sl_U r)\Big|_{U'\setminus\{0\}} \le \varepsilon.$$

*2) A mapping $r : X \to \mathbb{R}$ with $r(0) = 0$ is* HL*-small if and only if*

$$\exists U \in Nb_0(X)\ \forall \varepsilon > 0\ \exists \delta > 0\ :\ (sl_U r)\Big|_{\delta U\setminus\{0\}} \le \varepsilon.$$

The next simple lemma says that $U$-slope of a linear functional at each point is not greater then the supremum of this functional on $U$:

**Lemma.** *Let $x'$ be a linear functional on $X$, let $U$ be an absorbing balanced subset in $X$, and let $x'\Big|_U \le M$ $(M > 0)$. Then*

$$sl_U x' \le M$$

*(everywhere).*

To construct our example we use the following

**Basic lemma.** *Let $U_1 \supset U_2 \supset U_3 \supset \dots$ be a base of convex balanced closed neighborhoods of zero in $X$ (so that $X$ is a metrizable locally convex TVS). Let us suppose that there exists a countable family*

$$\{x^*_{nk}\}_{n,k\in\mathbb{N}}$$

*of continuous linear functionals $x^*_{nk}$ on $X$, such that the numbers*

$$\alpha_{nk} := \sup_{U_1} x^*_{nk}, \quad \beta_{nk} := \sup_{U_n} x^*_{nk}, \quad \gamma_{nk} := \sup_{U_{n+1}} x^*_{nk}$$

*satisfy the conditions*
*(i) all $\alpha_{nk}$; $\beta_{nk}$, $\gamma_{nk}$ are $> 0$;*
*(ii) $\alpha_{nk}$ and $\beta_{nk}$ does not depend on $k$:*

$$\alpha_{nk} = \alpha_n, \; \beta_{nk} = \beta_n;$$

*(iii) $\alpha_n \downarrow 0$ as $n \to \infty$;*
*(iv) $\forall n \in \mathbb{N}$*

$$\gamma_{nk} \downarrow 0 \quad as \quad k \to \infty.$$

*Then the function*

$$r := \bigvee_{n,k\in\mathbb{N}} ((x^*_{nk} - \gamma_{nk}) \vee 0)$$

*is* MS*-small, but is not* HL*-small (or equivalently, $r$ is* MS*-differentiable at 0 (with zero derivative), but is not* HL*-differentiable at 0).*

Here $\bigvee_\alpha f_\alpha$ denotes the supremum of a family of functions:

$$(\bigvee_\alpha f_\alpha)(x) := \sup_\alpha f_\alpha(x).$$

**Example.** *Let $H$ be a separable Hilbert space with an ortho-normal base $\{e_{nk}\}_{n,k\in\mathbb{N}}$, and let $A_i$ for each natural $i$ is a diagonal operator in $H$, given by the rule*

$$A_i(e_{nk}) = \begin{cases} e_{nk} & \textit{if} \quad i \neq n, \\ \frac{1}{k} e_{nk} & \textit{if} \quad i = n. \end{cases}$$

*Denote by $B$ the unit ball in $H$ and define by induction*

$$U_1 := B,$$

$$U_{i+1} := \frac{1}{2} A_i U_i.$$

*Consider continuous linear functionals* $x_{nk}^*$ *on* $H$*, corresponding to vectors* $n^{-1} e_{nk}$*:*

$$x_{nk}^*(x) := (n^{-1} e_{nk} | x);$$

*here* $(.|.)$ *denotes the scalar product in* $H$*. Put*

$$\gamma_{nk} := \sup_{U_{n+1}} x_{nk}^*$$

*and*

$$r := \bigvee_{n,k \in \mathbb{N}} ((x_{nk}^* - \gamma_{nk}) \vee 0).$$

*Then*
*1)* $\{U_i\}_{i \in \mathbb{N}}$ *is a base of neighborhoods of zero for a Hausdorff locally convex topology* $\tau$ *in* $H$*;*
*2)* $r$ *is* MS-*differentiable at zero, but is not* HL-*differentiable at zero;*
*3)* $r$ *is a continuous convex function on* $(H, \tau)$*.*

## 2.3 A partial result on the connection of b-differentiable and MS-differentiable continuous convex functions

The goal, which is not so far attained, is to prove that continuous convex b-differentiable function need not be MS-differentiable. At the moment I provide only the lemma that might be crucial in finding the right function and space to prove this.

**Lemma.** *Let* $X$ *be a Hausdorff locally convex topological vector space, let* $\{U_i\}_{i \in \mathbb{N}}$ *be a base of neigborhoods of zero, such that all* $U_i$ *are balanced, closed, ray-bounded and it holds* $U_{i+1} \subset \frac{1}{4} U_i$*. Let further*

$$\{x_{nk}\}_{n,k \in \mathbb{N}} \subset X$$

$$\{x_{nk}^*\}_{n,k \in \mathbb{N}} \subset X^*$$

*and*

$$\alpha_{nk} := \sup_{U_1} x_{nk}^*, \ \beta_{nk} := \sup_{U_n} x_{nk}^*, \ \gamma_{nk} := \sup_{U_k} x_{nk}^*,$$

$$f_{nk} := (x_{nk}^* - \gamma_{n,k+1}) \vee 0,$$

$$\big(so\ obviously\ \ f_{nk}|_{U_{k+1}} = 0\ \ \forall n\big) \tag{1}$$
$$f := \bigvee_{n,k\in\mathbb{N}} f_{nk},$$

*1) all* $\alpha_{nk} \leq 1,\ \alpha_{nk} \leq \alpha_n \searrow 0$ *as* $n \to \infty$

*2)* $\forall n\ \exists\ \beta_n > 0\ \forall k \vdots \beta_{nk} \geq \beta_n$,

*3)* $\forall n\ \forall h \in X \vdots x^*_{nk}(h) \to 0$ *as* $k \to \infty$,

*4)* $\forall n, k \vdots x_{nk} \in U_k$,

*5)* $\forall n, k \vdots x^*_{nk}(x_{nk}) \geq \frac{1}{2}\gamma_{nk}$,

*6)* $\forall n, k \vdots \mu U_n(x_{nk}) \leq \frac{\gamma_{nk}}{\beta nk}$.

*Then*

*a) f is continuous and convex,*
*b) f is MB-differentiable,*
*c) f is not MS-differentiable.*

The problem, which is remained so far unsolved, is to construct a Montel Hausdorff locally convex metrizable space, for which there exist $x_{nk}$ and $x^*_{nk}$ satisfying the conditions of the lemma (for Montel spaces MB-differentiability implies b-differentiability, see the diagram in 2.3).

# 3 Publications

1. *Konderla T.*: Differentiation of Continuous Convex Functions, ISSN 0001-4346, Mathematical Notes, 2012, Vol. 91, No. 1, pp. 65-68.

2. *Averbuch V.I., Konderla T.*: Continuous Convex MS-differentiable Function Need Not Be HL-differentiable, ISSN 0001-4346, Mathematical Notes, 2012, Vol. 91, No. 2, pp. 153-160. (Of this article I am author of 50 percent)

# 4 Presentations

During my study I presented several times my status of research and the articles on the seminars of prof. Smítal and prof. Engliš.

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