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Some aspects of chaos in discrete dynamical systems

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Contents

1 Indroduction

The Thesis consists of three independent papers, [1],[2] and [3] (one is published, one is accepted for publication and one is submitted), which study some aspects of nonchaotic and chaotic behavior in discrete dynamical systems. Throughout the Thesis we consider only discrete dynamical systems generated by continuous maps of a compact metric space, or especially of the unit interval, into itself.

The first part of the Thesis concerns with the iterative stability of continuous functions of the interval with respect to small perturbations. More precisely, we study the continuity structure of the map $CR : (C(I), ||\cdot||) \to (K, \rho_H)$, which maps $f \in C(I)$ to the set $CR(f)$ of chain recurrent points of f.

In the next two parts we study chaotic behavior of discrete dynamical systems. The second part solves a problem concerning iteration invariants. Particularly, we show that if f is a *DC3* continuous map of a compact metric space then also f^N is *DC3*, for every positive integer N.

In the last part we study chaos in nonautonomous discrete dynamical systems generated by sequences of continuous self-maps of the unit interval. Especially we investigate connections between chaotic behavior of the nonautonomous discrete dynamical system and chaotic behavior of its limit function. We show that even the full Lebesgue measure of a distributionally scrambled set of the nonautonomous system does not guarantee the existence of distributional chaos of the limit map and, conversely, that there is a nonautonomous system with arbitrarily small distributionally scrambled set that converges to a map distributionally chaotic a.e.

2 Basic terminology and notation

Let (X, ρ) be a compact metric space, $I = [0, 1]$ the unit interval, $\mathbb{N}$ the set of all positive integers and $C(X)$ the class of continuous self-maps of X. For any $f \in \mathcal{C}(X)$ the pair (X, f) is a *discrete (autonomous) dynamical system* and, for any $n \in \mathbb{N}$, by $f^n(x)$ we denote the nth iteration of x under f. The *trajectory* of an $x \in X$ is the sequence $\{x_n\}_{n=0}^{\infty}$, where $x_0 = x$ and $x_n = f^n(x)$. The set of all limit points of the trajectory of the $x \in X$ is called the *ω-limit set* of x and it is denoted by $\omega(x, f)$. A point $x \in X$ is *periodic* of period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^j(x) \neq x$ for $j = 1, 2, \cdots, n-1$. By $P(f)$, $\omega(f)$, $NW(f)$ and $CR(f)$ we denote the set of periodic points of f, the union of ω-limit sets of f, the set of non-wandering points of f, and the set of chain recurrent points of f, respectively. Recall that $x \in NW(f)$ if, for every neighborhood U of x, there is an $s \in \mathbb{N}$ such that $f^s(U) \cap U \neq \emptyset$, and $x \in CR(f)$ if there is an ϵ-chain from x to itself for any $\epsilon > 0$. An *ϵ-chain* from x to y with respect to a function f is a finite set of points $\{x_0, x_1, \cdots, x_n\}$ in X with $x = x_0$, $y = x_n$ and $\rho(f(x_{k-1}), x_k) < \epsilon$. A point $p \in I$ is an *essential periodic point* of a map $f \in C(I)$ of period $n \in \mathbb{N}$ if, in every neighborhood of p there are points x, y with $f^n(x) > x$ and $f^n(y) < y$. A periodic orbit A of f with period $n \in \mathbb{N}$ is *p-stable* if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that every $g \in C(I)$ with $\|f - g\| < \delta$ has a periodic orbit B of period n satisfying $\rho_H(A, B) < \varepsilon$, where ρ_H denotes the Hausdorff metric. We denote by $S_0(f)$ and $S(f)$ the union of essential periodic points, and p-stable periodic orbits of f, respectively. Recall that, for every $f \in C(I)$ and $j \geq 0$, $f^j(S_0(f)) \subseteq S(f)$, see [SmSt]. Let us note that the sets $\omega(f)$, $NW(f)$ and $CR(f)$ are closed in I and the relationship between them is as follows

$$S(f) \subseteq P(f) \subseteq \overline{P(f)} \subseteq \omega(f) \subseteq NW(f) \subseteq CR(f),$$

where $\overline{P(f)}$ is the closure of the set of periodic points.

We say that a map f is *topologically transitive* if for any nonempty open sets $A, B \subset X$ there is an $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \emptyset$, a map f is *bitransitive* if f^2 is transitive. A function $f \in \mathcal{C}(X)$ is *conjugate* to $g \in \mathcal{C}(X)$ if there is a homeomorphism $h \in \mathcal{C}(X)$ such that $f = h^{-1} \circ g \circ h$.

The pair of points x, y is called a *Li-Yorke pair*, if

$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > 0, \text{ and } \liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0.$$

A set containing at least two points is called a *LY-scrambled set*, if any pair of its distinct points forms a Li-Yorke pair. A map $f \in \mathcal{C}(X)$ is *chaotic in the sense of Li and Yorke*, briefly *LYC*, if there exists an uncountable LY-scrambled set $S \subset X$. Moreover it is *extremely LYC*, if for any distinct points x and y from an uncountable LY-scrambled set S,

$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) = \operatorname{diam}(X), \text{ and } \liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0.$$

For any $n \in \mathbb{N}$, points $x, y \in X$ and $t \in \mathbb{R}$, $0 < t \leq \text{diam}\,(X)$, we define

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n; \rho(f^j(x), f^j(y)) < t\}$$

and

$$\Phi^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n; \rho(f^j(x), f^j(y)) < t\}.$$

The functions $\Phi_{xy}, \Phi^*_{xy} : (0, \text{diam}\ X] \to [0, 1]$ are the *lower* and the *upper distribution functions* of x, y, respectively. There exist three types of distributional chaos, *DC1 - DC3*. The pair of points $x, y \in X$ such that

$$\begin{array}{ll} \Phi^*_{xy} \equiv 1 \text{ and } \Phi_{xy}(t) = 0 \text{ for some } t > 0, \text{or} & (D1) \\ \Phi^*_{xy} \equiv 1 \text{ and } \Phi_{xy}(t) < \Phi^*_{xy}(t) \ \text{ for all } t \text{ in an interval, or} & (D2) \\ \Phi_{xy}(t) < \Phi^*_{xy}(t) \ \text{ for all } t \text{ in an interval,} & (D3) \end{array}$$

is called *distributionally chaotic of type 1-3*, briefly *D1, D2*, or *D3*, respectively. A set containing at least two points is called *a distributionally scrambled set of type 1-3* if any pair of its distinct points is distributionally chaotic of type 1-3, respectively. A map $f \in \mathcal{C}(X)$ is *distributionally chaotic of type 1-3*, briefly *DC1, DC2*, or *DC3*, if there exists an uncountable distributionally scrambled set $S \subset X$ of type 1-3, respectively. Let us recall that *DC1* is the original version of distributional chaos and it was introduced for interval maps in [SSm]. Later there were introduced two weaker versions of distributional chaos *DC2* and *DC3*, see [BSS]. Directly from the definitions it follows that *DC1* implies *DC2* and *DC2* implies *DC3*. Another terminology and notation will be defined in the next parts in the appropriate places.

3 Chain recurrence

In this section we will work with two metric spaces. The first space is $(C(I), ||\cdot||)$ with the metric of uniform convergence, the second one, (K, ρ_H), is the class of nonempty closed subsets K of I with the Hausdorff metric ρ_H. Recall that $\rho_H(E, F)$ is the minimal $\varepsilon \geq 0$ such that $B_\varepsilon(E) \supseteq F$ and $B_\varepsilon(F) \supseteq E$, where $B_\varepsilon(A)$ denotes the closed ε-neighborhood of the set A. By C, ω, NW and CR we denote the map

$(C(I), ||\cdot||) \to (K, \rho_H)$ which takes $f \in C(I)$ to $C(f)$, $\omega(f)$, $NW(f)$, and $CR(f)$, respectively.

In recent years, there has been devoted attention to the study of continuity and iterative stability of continuous self-maps of the unit interval with respect to small perturbations. At the Twentieth Summer Symposium in Real Analysis, A. M. Bruckner [Br] posed, among others, a question, how the slight changes in the function affect the set of ω-limit points and the collection of ω-limit sets. In general, only small perturbations could affect dramatically both these sets. T. H. Steele [St] has solved one of these problems by characterizing the points of continuity of the map ω as the maps $g \in C(I)$ whose p-stable periodic orbits are dense in $CR(g)$. Recently, J. Smítal and T. H. Steele [SmSt] have proved that similar results are true for the maps C and NW. We continue in their work and characterize those functions $f \in \mathcal{C}(I)$ at which the map $CR : (C(I), ||\cdot||) \to (K, \rho_H)$, $f \mapsto CR(f)$, is continuous.

Theorem A. (See [1].) *The map CR is continuous at $g \in C(I)$ if and only if $\overline{S(g)} = CR(g)$.*

We perceive continuity of map CR at g as a form of stability of the set of chain recurrent points at g. In light of this knowledge we can say as a consequence of Theorem A and previous results that for a continuous map g of the unit interval, either all of the maps C, ω, NW and CR are continuous at g, or all these maps are discontinuous at g.

4 Iteration invariants for distributional chaos

Recently, Li [Li] proved that, for $f \in C(X)$, both *DC1* and *DC2* are iteration invariants, i.e., for any $N \in \mathbb{N}$, f is *DC1* (resp. *DC2*) if and only if f^N is also *DC1* (resp. *DC2*). The natural question was if it is true also for *DC3*. Li presented two examples supporting the hypothesis that *DC3* is the iteration invariant. More precisely, the question was whether the following two implications are true
(1) If f is *DC3*, then f^N is *DC3*, for every $N \in \mathbb{N}$,
(2) If f^N is *DC3* for some $N \in \mathbb{N}$, then f is *DC3*.
We proved the first of them.

Theorem B. (See [2].) *Let f be a continuous map of a compact metric space X. If f is DC3 then, for every $N \in \mathbb{N}$, f^N is DC3.*

Furthermore we showed that if we consider two points $x, y \in X$, which generate *DC3* for the N-th iteration of the map f, then, in general, these two points do not generate *DC3* for f. This obviously does not disprove the second implication, that $f^N \in DC3$ implies $f \in DC3$. It only demonstrates, that if this result is true, the argument would not be easy. Let us also note that if we would like to disprove it, we would need to find an example of a map, which is *DC3* but not *DC2*. As we know, there are known only five examples of maps $f \in \mathcal{C}(X)$ which are $DC3$ but not $DC2$, see [BSS], [Op], [PS], [Li], [SM]. However, it seems that none of these examples can be modified in order to disprove the implication $f^N \in DC3 \Rightarrow f \in DC3$. We conjecture that *DC3* is an iteration invariant.

5 Chaos in nonautonomous discrete dynamical systems

The last section deals with nonautonomous discrete dynamical systems. A *nonautonomous discrete dynamical system* is a pair $(X, f_{1,\infty})$, where $f_{1,\infty} \equiv \{f_n\}_{n \geq 1}$ is a sequence of continuous maps $f_n \in C(I)$. The *trajectory* of an $x \in X$ in this system is the sequence $\{x_n\}_{n=0}^{\infty}$, where $x_0 = x$ and $x_n = f_n \circ f_{n-1} \circ \cdots \circ f_1(x)$. If $f_n = f$ for every $n \in \mathbb{N}$ then obviously nonautonomous dynamical system $(X, f_{1,\infty})$ becomes the autonomous one (X, f). We consider a particular case of nonautonomous discrete dynamical systems such that the sequence $f_{1,\infty}$ converges uniformly to a continuous map f and moreover, to avoid some pathological cases, f and all maps in $f_{1,\infty}$ are surjective (without this assumption, e.g., the single constant function added to $f_{1,\infty}$ can destroy even a very complex behavior of $f_{1,\infty}$).

The study of the dynamical behavior of nonautonomous dynamical systems is recently very intensive, because it appears in almost all fields, where the dynamical progress is studied. One of the natural questions, whether the simplicity of the limit function f implies the simplicity of the nonautonomous system $(I, f_{1,\infty})$, was, in the case of Li-Yorke chaos, already disproved in [FPS]. Later it was shown that the positive topological entropy of f implies Li-Yorke chaos of $(I, f_{1,\infty})$, see [Ca]. There was also posed a question, if, in general, the chaoticity of f always implies the chaoticity of $(I, f_{1,\infty})$.

Another interesting question to prove or disprove is whether the full Lebesgue measure of a scrambled set of $(I, f_{1,\infty})$ implies chaoticity of the limit function f, and conversely, if the chaoticity a.e. of map f guarantees the chaoticity of $(I, f_{1,\infty})$. We have found two examples of nonautonomous systems which show that, in general, there is no connection between the "size" of the scrambled sets for $(I, f_{1,\infty})$ and for its limit function f.

Theorem C. (See [3].) *There is a surjective nonautonomous system $(I, f_{1,\infty})$ such that, for every $n \in \mathbb{N}$, $(I, f_{n,\infty})$ is distributionally chaotic almost everywhere* (*the scrambled set is the whole interval* $(0, 1)$)*, and such that $(I, f_{1,\infty})$ uniformly converges to a nonchaotic map $f \in C(I)$.*

Theorem D. (See [3].) *There is a surjective nonautonomous system $(I, f_{1,\infty})$ converging uniformly to a map $f \in \mathcal{C}(I)$, and such that both (I, f) and $(I, f_{1,\infty})$ are LYC. Moreover, (I, f) has an (extremely) scrambled set S of full Lebesgue measure, but every Li-Yorke scrambled set of $(I, f_{1,\infty})$ has zero Lebesgue measure.*

The idea of the proof is based on the fact, that every continuous bitransitive map of the interval is conjugate to a map extremely chaotic in the sence of Li and Yorke a.e., see [Ba]. We construct a nonautonomous system $(I, f_{1,\infty})$, such that any of its Li-Yorke scrambled sets has zero Lebesgue measure, and it converges uniformly to the tent map τ, which is bitransitive. Finally, let us remark, that a similar assertion holds for distributional chaos, too: Every continuous bitransitive map of the interval is conjugate to a map distributionally chaotic a.e., see [BaS]. Based on this fact we can state the following theorem.

Theorem E. (See [3].) *There is a nonautonomous system $(I, f_{1,\infty})$ such that $f_{1,\infty}$ is a sequence of surjective maps in $\mathcal{C}(I)$ converging uniformly to a map $f \in \mathcal{C}(I)$, and such that both (I, f) and $(I, f_{1,\infty})$ are DC1. Moreover, (I, f) has a DC1-scrambled set S of full Lebesgue measure, but every DC1-scrambled set of $(I, f_{1,\infty})$ has zero Lebesgue measure.*

6 Publications concerning the Thesis

[1] Dvořáková J., *Stability of chain recurrent points of continuous maps on interval.* J. Difference Equ. Appl., DOI:10.1080/10236198.2010.545057. (IF 1.0)

[2] Dvořáková J., *On a problem of iteration invariants for distributional chaos.* Commun. Nonlinear. Sci. Numer. Simulat. 17 (2012), 785–787. (IF 2.7)

[3] Dvořáková J., *Chaos in nonautonomous discrete dynamical systems.* Submitted.

7 Conferences

[4] Visegrad Conference Dymanical Systems, High Tatras 2007, Štrbské Pleso, Slovakia, June 17–23, 2007.

[5] VII Iberoamerican Conference on Topology and its Applications, Valencia, Spain, June 25–28, 2008.
Talk: "A characterization of omega-limit sets of C^2 functions on the interval".

[6] 12th Czech-Slovak Workshop on Discrete Dynamical Systems, Pustevny, Czech Republic, September 14–20, 2008.

[7] 13th Czech-Slovak Workshop on Discrete Dynamical Systems, Jeseníky mountains, Czech Republic, September 6–13, 2009.

[8] International Conference on Difference Equations and Applications, Estoril, Portugal, October 19–23, 2009.
Talk: "Stability of chain recurrent points of a continuous map of the interval".

[9] 14th Czech-Slovak-Spanish Workshop on Discrete Dynamical Systems, La Manga del Mar Menor, Spain, September 20–24, 2010.
Talk: "On minimal points of commuting maps".

[10] 15th Czech-Slovak Workshop on Discrete Dynamical Systems, Banská Bystrica, Slovakia, June 27 – July 3, 2011.
Talk: "On a problem of iteration invariants for distributional chaos".

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