Report

The habilitation thesis submitted by Mgr. Krysztof Ciepliński, Ph.D has the title *Conjugacy equation in dimension one and its applications in iteration theory.* The conjugacy equation in general is the equation

$$\Phi \circ F_t = G_t \circ \Phi \tag{1}$$

where for a given set X and a given set M the families $(F_t)_{t\in M}$ and $(G_t)_{t\in M}$ have the property that F_t and G_t map X into itself and where one looks for solutions $\Phi\colon X \to X$.

"Dimension one" is interpreted as the case where X is either an open interval in $\mathbb{R}$ or $X = \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the unit circle. "Conjugacy equation" comes from the original problem of finding bijective solutions Φ, when (1) may be rewritten as $F_t = \Phi^{-1} \circ G_t \circ \Phi$.

It is obvious that in this generality even in the restriction to dimension one (almost) nothing can be said about (1) and its solution. The situation becomes more interesting when certain regularity conditions are imposed on the given data as well as on the solutions Φ.

A special case of (1) is

$$\Phi \circ F_t = c(t)\Phi,\ t \in M, \tag{2}$$

where $M = \mathbb{S}^1$ and $c(t) \in \mathbb{S}^1$ for all $t \in M$, or even more specific (M a singleton, $s \in \mathbb{S}^1$)

$$\Phi \circ F = s\Phi. \tag{3}$$

Chapter 1 of the thesis gives an introduction to the topic and starts with citing the (six) papers forming the habilitation thesis. These papers have been published in the period of years between 2002 and 2011. Three of them are written jointly with Marek Cezary Zdun.

Chapter 2 contains the framework necessary to present the results to come. The most important facts here are certain results on continuous self-mappings of $\mathbb{S}^1$: *For each continuous function $F\colon \mathbb{S}^1 \to \mathbb{S}^1$ there is some continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and some integer k such that*

$$F(\exp(2\pi i x)) = \exp(2\pi i f(x)), \quad x \in \mathbb{R} \tag{4}$$

and

$$f(x+1) = f(x) + k, \quad x \in \mathbb{R}. \tag{5}$$

Up to a translation by some integer this function f is uniquely determined by F and called the *lift* of F. The integer k satisfying (5) is called the *degree* of F and denoted by $\deg F$.

F is a homeomorphism if and only its lift f is a homeomorphism. Moreover $\deg F \in \{\pm 1\}$ if F is a homeomorphism. Homeomorphisms F with $\deg F = +1$ are called *orientation-preserving* and *orientation-reversing* otherwise. The lifts of orientation-preserving (-reversing) homeomorphisms are monotonically increasing (decreasing) homeomorphisms.

Moreover, given an orientation-preserving homeomorphism F of $\mathbb{S}^1$ with lift f, it turns out that for any $x \in \mathbb{R}$ the limit $\lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1$ exists and is independent of x. So this number only depends on F. It is called the *rotation number* of F and denoted by $\alpha(F)$.

A family $(F_v)_{v\in V}$ of homeomorphisms $F_v\colon X \to X$, where X is an open interval in $\mathbb{R}$ or $X = \mathbb{S}^1$ is called an *iteration group* if V is a non-trivial 2-divisible abelian group and if $F_{u+v} = F_u \circ F_v$ for all $u, v \in V$.

Chapter 3 contains the main results of the thesis. Theorem 3.1.1 says that $\Phi \circ F = G \circ \Phi$ for homeomorphisms F, G of $\mathbb{S}^1$ and continuous Φ implies $\alpha(G) = \alpha(F) \cdot \deg \Phi \pmod 1$. Thus this theorem generalizes earlier results where Φ was assumed to be a homeomorphism. For the special case $G(z) = sz$, $s \in \mathbb{S}^1$, this implies that

$$s = \exp(2\pi i \alpha(F) \deg \Phi).$$

The author also gives an extension of a result by POINCARÉ:

> *For any orientation-preserving homeomorphism $F\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\alpha(F)$ is irrational, there is a unique continuous function $\Phi_F\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Phi_F \circ F = \exp(2\pi i \alpha(F))\Phi_F$ and $\Phi_F(1) = 1$. Moreover Φ_F is an orientation-preserving homeomorphism if and only if the* limit set *of F, L_F, is equal to $\mathbb{S}^1$.*

This result is used to solve (2) (and (1)) under certain conditions on the family $(F_t)_{t\in M}$ (and the family $(G_t)_{t\in M}$). The limit sets L_{F_t} and L_{G_t} are used to characterize the situations when Φ as a continuous function is determined (almost) uniquely and when Φ does depend on an arbitrary function.

Theorem 3.2.2 gives conditions on $(F_t)_{t\in M}$ such that (2) holds true for some function $c\colon M \to \mathbb{S}^1$ and some continuous Φ.

Among many other topics all n-th roots of $\mathrm{id}_{\mathbb{S}^1}$ in the class of orientation-preserving homeomorphisms are constructed.

A great part of the habilitation thesis deals with the task to find homeomorphisms Φ in the equations 1–3 when the families (F_t) and (G_t) are iteration groups.

Summarizing (and omitting to mention many other results of the thesis) I want to express my opinion that the work gives a very detailed description of some parts of the theory of dynamical systems in dimension one. The topics considered are mainly connected with the Schröder equation and with iteration groups and semigroups.

The author definitely shows his skills in the area and I highly recommend his promotion to the rank of *Docent*.

With kindest regards

Jens Schwaiger