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Distributional chaos in compact metric spaces

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Distribuční chaos v kompaktních metrických prostorech

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Contents

1. Introduction

The term "chaos" in connection with a function was introduced by Li and Yorke in 1975, since then several different definitions of what it means for a function to be chaotic have been proposed. One could say "as many authors, as many definitions of chaos"; most of them are based on the idea of unpredictability of the behavior of trajectories or sensitive dependence on initial conditions. The idea of chaos emerged from experiments in physics. Physicists expressed their opinion of what mathematical property could describe chaotic behavior and then mathematicians began using the word "chaos" as a label for this property. More recently, people from other fields have started to think about chaos - in computer science, chaos is connected to computational complexity. Biology and economics have also produced their own concepts of disorder. This creates some confusion in contemporary mathematical literature and therefore the word "chaos" should always be understood in the right context.

It often occurs that in a restricted class of topological dynamical systems several chaotic properties or definitions of chaos coincide, or one implies another, while this is false when looking at the whole category of dynamical systems - among interval maps distributional chaos and positive topological entropy are equivalent properties, something which is false in general. That each smaller class of dynamical systems has its own theory of chaos is important. The picture of chaos changes completely when one extends the scope from interval maps to the whole field of topological dynamics.

The reader should be aware that many topological properties are called chaotic, because they evoke a feeling of chaos in some way, but none of them can be proved to be chaotic because there is no general definition of chaos and there will not be any. The definition of distributional chaos meets my expectation of complex and unpredictable behavior. Most results about distributional chaos hold only in restricted classes of maps acting on an interval, graph, square etc. I decided to consider maps on general compact metric space and examine distributional chaos in this more universal setting. The scope of this thesis is still limited to topological dynamics, stochastic chaos is not taken into account, nor is the measure-theoretic approach, although their relation to distributional chaos is part of the current research.

2. Distributional chaos

Chaotic (or scrambled) pairs were first brought into the theory of dynamical systems by Li and Yorke in [1], who studied pairs of points with the property that their orbits neither approach each other asymptotically, nor do they eventually separate from each other by any fixed positive distance. A dynamical system is called chaotic in the sense of Li-Yorke if it possesses an uncountable scrambled set such that each pair of its distinct points is chaotic.

It was a long standing problem, what are the relations between positive topological entropy and the existence of Li-Yorke chaos. Topological entropy is a measure of the complexity of a dynamical system - for systems with positive topological entropy, the number of distinguishable orbits grows exponentially with time. A theorem by Misiurewicz [4] which characterizes positive topological entropy of interval maps in terms of topological horseshoes provided a tool for proving that positive topological entropy implies the existence of an uncountable scrambled set [8]. Since Xiong [5] and Smítal [11] constructed some interval maps with zero topological entropy which are Li-Yorke chaotic, Li-Yorke chaos was found to be a necessary but not sufficient condition for positivity of topological entropy.

Schweizer and Smítal [2] introduced the related concept of a distributionally chaotic

pair, which means, roughly speaking, that the statistical distribution of distances between the orbits does not converge. They discovered that the existence of a single distributionally chaotic pair is equivalent to positivity of topological entropy when restricted to the interval case. This fact, combined with the characterization of maps with positive entropy by S. Li in [10], shows that the existence of a distributionally chaotic pair forces a very strong chaotic behavior. In particular, distributional chaos, positive topological entropy, ω-chaos and chaos in the sense of Devaney are all equivalent properties for interval maps. The equivalence of different kinds of chaos is no longer valid for maps acting on a general compact metric space. Later the definition of distributional chaos was split into three versions of distributional chaos (briefly, DC1 – DC3), equivalent for the interval case but distinct for a general dynamical system. Relations between them and their properties were investigated by many authors, see e.g. [3, 6]. One can easily see from the definitions that DC1 implies DC2 and DC2 implies DC3. On the other hand, there are examples which show that DC1 is stronger than DC2 and DC2 is stronger than DC3. It is also obvious that either DC1 or DC2 implies Li-Yorke chaos. For maps acting on a general compact metric space, positive topological entropy implies DC2 by [7].
Two more chaotic properties of interval maps are equivalent to the positivity of topological entropy - invariant and multivariant chaos. By invariant chaos we mean existence of an invariant chaotic set (see [12]), by multivariant chaos we mean existence of scrambled n-tuples. Existence of one scrambled triple for $f \in \mathcal{C}(I)$ implies f having positive topological entropy by [9]. In particular, existence of a scrambled triple for f implies existence of an uncountable chaotic set (in the sense of pairs) and therefore scrambled triples are always accompanied by scrambled pairs.

3. Structure of the thesis

The thesis consists of four separate articles unified by the same subject - distributional chaos. In article (A), we state a sufficient condition for invariant distributional chaos in general compact metric spaces. This result has already been improved in [13], where authors showed that this sufficient condition is equivalent to the specification property. In article (B), we investigate the relation between distributional chaos and existence of scrambled triples. We show that a general dynamical system can possess a distributionally chaotic set (in the sense of pairs) but no scrambled triples. Article (C) examines whether the distributional chaos is preserved under semiconjugacy. It is proved in [3] that DC3 does not imply chaos in the sense of Li and Yorke and it is not preserved under topological conjugacy. Hence the definition of DC3 was strengthened in such a way that it is preserved under conjugacy and implies Li-Yorke chaos, but is still weaker than DC2 in [14] – the new definition was denoted by DC2$\frac{1}{2}$. We show in (D) that DC2$\frac{1}{2}$ is also iteration invariant.

4. Terminology

Let (X, d) be a non-empty compact metric space. A pair (X, f), where f is a continuous self-map acting on X, is called a *topological dynamical system*. We define the *forward orbit* of x, denoted by $Orb_f^+(x)$ as the set $\{f^n(x) : n \geq 0\}$, where f^n is the nth iterate of f. A non-empty closed invariant subset $Y \subset X$ defines naturally a subsystem (Y, f) of (X, f). For $n \geq 2$, we denote by $(X^n, f^{(n)})$ the product system $(X \times X \times ... \times X, f \times f \times ... \times f)$ and put $\Delta^{(n)} = \{(x_1, x_2, ..., x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}$.
By a *perfect set* we mean a nonempty compact set without isolated points. A *Cantor set* is a perfect and totally disconnected set. A set $D \subset X$ is *invariant* if

$f(D) \subset D$. A *Mycielski set* is defined as a countable union of Cantor sets.
A continuous map $\pi : X \to Y$ is called a *semiconjugacy* between (Y, f) and (X, F) if π is surjective and $\pi \circ F = f \circ \pi$. In this case, we can say that (Y, f) is a factor of the system (X, F), equivalently (X, F) is an extension of the system (Y, f). If π is also a homeomorphism, we say π is a *conjugacy* and (X, F) is conjugated to (Y, f).
Property P is *preserved under conjugacy* if, for any dynamical system (X, F) and for any (Y, f) conjugate to (X, F), (X, F) has P if and only if (Y, f) has P. Property P is an *iteration invariant* if, for any dynamical system (X, f) and $n \in \mathbb{N}$, (X, f) has P if and only if (X, f^n) has P. By I we denote the compact unit interval $[0, 1]$. By the k-dimensional unit cube we mean the set I^k, where $k \geq 1$. The Lebesgue measure on I^k will be denoted by λ.
A point $x \in X$ is said to be periodic with period $n > 1$ if $f^n(x) = x$ but $f^i(x) \neq x$, for all $0 < i < n$. A point x is said to be fixed if $f(x) = x$.
We say that a pair $(x, y) \in X^2$ is *asymptotic* if $\lim_{n\to\infty} d(f^i(x), f^i(y)) = 0$ and *eventually equal* if there is $j \in \mathbb{N}$ such that $f^j(x) = f^j(y)$. We say that a pair (x, y) is *proximal* if $\liminf_{n\to\infty} d(f^i(x), f^i(y)) = 0$ (otherwise we say (x, y) is *distal*). A pair of points is *Li-Yorke scrambled* simply if it is proximal but not asymptotic.

We proceed with the definition of distribution functions for a pair $(x_1, x_2) \in X^2$ whose value at δ may be interpreted as lower and upper asymptotic densities of times when the distance between the trajectories of x_1 and x_2 is less than δ.

Definition 1. *For a pair (x_1, x_2) of points in X, define the* lower distribution function *generated by f as*

$$\Phi_{(x_1,x_2)}(\delta) = \liminf_{m\to\infty} \frac{1}{m} \#\{0 \leq k \leq m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

and the upper distribution function *as*

$$\Phi^*_{(x_1,x_2)}(\delta) = \limsup_{m\to\infty} \frac{1}{m} \#\{0 \leq k \leq m; d(f^k(x_1), f^k(x_2)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.
A pair $(x_1, x_2) \in X^2$ is called distributionally chaotic of type 1 (briefly DC1) *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \text{ for every } 0 < \delta \leq \text{diam } X, \quad \Phi_{(x_1,x_2)}(\epsilon) = 0, \text{ for some } 0 < \epsilon \leq \text{diam } X,$$

distributionally chaotic of type 2 (briefly DC2) *if*

$$\Phi^*_{(x_1,x_2)}(\delta) = 1, \text{ for every } 0 < \delta \leq \text{diam } X, \quad \Phi_{(x_1,x_2)}(\epsilon) < 1, \text{ for some } 0 < \epsilon \leq \text{diam } X,$$

distributionally chaotic of type $2\frac{1}{2}$ (briefly DC$2\frac{1}{2}$) *if there exist numbers $c, q > 0$ such that*

$$\Phi_{(x_1,x_2)}(\delta) < c < \Phi^*_{(x_1,x_2)}(\delta), \text{ for every } 0 < \delta \leq q,$$

distributionally chaotic of type 3 (briefly DC3) *if*

$$\Phi_{(x_1,x_2)}(\delta) < \Phi^*_{(x_1,x_2)}(\delta), \text{ for every } \delta \in (a, b), \text{ where } 0 \leq a < b \leq \text{diam } X.$$

We can define both distribution functions at 0 as the limit $\Phi_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+} \Phi_{(x_1,x_2)}(\delta)$ and $\Phi^*_{(x_1,x_2)}(0) = \lim_{\delta\to 0^+} \Phi^*_{(x_1,x_2)}(\delta)$. Then (x_1, x_2) being DC1 is equivalent to

$$\Phi^*_{(x_1,x_2)}(0) = 1, \quad \Phi_{(x_1,x_2)}(\epsilon) = 0, \text{ for some } 0 < \epsilon \leq \text{diam } X;$$

DC2 is equivalent to

$$\Phi^*_{(x_1,x_2)}(0) = 1, \quad \Phi_{(x_1,x_2)}(0) < 1;$$

DC$2\frac{1}{2}$ is equivalent to

$$\Phi_{(x_1,x_2)}(0) < \Phi^*_{(x_1,x_2)}(0).$$

A subset S of X is called *distributionally scrambled* of type i if every pair of distinct points in S is distributionally chaotic of type i. There are two ways to define distributional chaos - either as the existence of at least one distributionally scrambled pair or the existence of an uncountable distributionally scrambled set. We say that the dynamical system (X, f) is *distributionally chaotic* of type i (DCi for short), where $i = 1, 2, 2\frac{1}{2}, 3$, if there is at least one distributionally scrambled pair of type i in X. When we use the other way of defining distributional chaos, we will emphasize the fact that an uncountable distributionally scrambled set is concerned. The dynamical system is strictly DCi if it possesses no distributionally chaotic pairs of types smaller than i.
The concept of scrambled pairs can be generalized to obtain scrambled n-tuples:

Definition 2. *A tuple* $(x_1, x_2, ..., x_n) \in X^n$ *is called* scrambled *if*

$$\liminf_{k\to\infty} \max_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) = 0 \tag{1}$$

and

$$\limsup_{k\to\infty} \min_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) > 0. \tag{2}$$

A subset S of X is called *n-scrambled* if every n-tuple $(x_1, x_2, ..., x_n) \in S^n \setminus \Delta^{(n)}$ is scrambled. The system (X, f) is called *n-chaotic* if there exists an uncountable n-scrambled set.

We define distributionally scrambled n-tuples only in the strongest sense of distributional chaos (usually called DC1):

Definition 3. *For an n-tuple* $(x_1, x_2, ..., x_n)$ *of points in X, define the* lower distribution function *generated by f as*

$$\Phi_{(x_1,x_2,...,x_n)}(\delta) = \liminf_{m\to\infty} \frac{1}{m}\#\{0 < k < m; \min_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) < \delta\},$$

and the upper distributional function *as*

$$\Phi^*_{(x_1,x_2,...,x_n)}(\delta) = \limsup_{m\to\infty} \frac{1}{m}\#\{0 < k < m; \max_{1\le i<j\le n} d(f^k(x_i), f^k(x_j)) < \delta\},$$

where $\#A$ denotes the cardinality of the set A.
A tuple $(x_1, x_2, ..., x_n) \in X^n$ *is called* distributionally scrambled *if*

$$\Phi^*_{(x_1,x_2,...,x_n)} \equiv 1 \text{ and } \Phi_{(x_1,x_2,...,x_n)}(\delta) = 0, \text{ for some } 0 < \delta \le \text{diam } X.$$

A subset S of X is called *distributionally n-scrambled* if every n-tuple $(x_1, x_2, ..., x_n) \in S^n \setminus \Delta^{(n)}$ is distributionally scrambled.

A subset S of X is called *extremal distributionally n-scrambled* if every n-tuple $(x_1, x_2, ..., x_n) \in S^n \setminus \Delta^{(n)}$ is a distributionally scrambled with $\Phi_{(x_1,x_2,...,x_n)}(\delta) = 0$, for any $0 < \delta < \text{diam } X$.

Definition 4. *A function f acting on X has the strong specification property (briefly SSP) if, for any $\delta > 0$, there is a positive integer $K(\delta)$ such that, for any integer $s \ge 2$, any set $\{y_1, y_2, ..., y_s\}$ of s points of X, and any sequence $0 = j_1 \le k_1 < j_2 \le k_2 < j_s \le k_s$ of $2s$ integers with $j_{m+1} - k_m \ge K(\delta)$ for $m = 0, ..., s-1$, there is a point x in X such that*

$$f^n(x) = x, \qquad (P)$$

where $n = K(\delta) + k_s$ and for each positive integer $m \le s$ and all integers i with $j_m \le i \le k_m$,

$$d(f^i(x), f^i(y_m)) < \delta.$$

A function f has the weak specification property (briefly WSP) if f fulfills the above-mentioned conditions only for the special case $s = 2$. Because we will not need the full force of this notion to obtain our results, we can omit the periodicity condition (P) and denote this version of the specification property by WSP*.

5. Main results

A. **Distributionally scrambled invariant sets in a compact metric space.**

Theorem 1. *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with WSP* which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a point $x \in X$ such that $(f^i(x), f^j(x))$ is a DC1 pair for all $i \neq j$, i.e., the forward orbit of x is a distributionally scrambled set of type 1.*

Remark Later, authors in [13] found that the condition of having infinitely many periodic points can be omitted. The assumption about a fixed point is natural - if x belongs to an invariant scrambled set, then $(x, f(x))$ is proximal. By the compactness of X there is an increasing sequence k_i and a point $p \in X$ such that $\lim_{i\to\infty} f^{k_i}(x) = p$ and simultaneously $\lim_{i\to\infty} d(f^{k_i}(x), f^{k_i}(f(x))) = 0$, which by continuity of f implies that $f(p) = p$.

In the previous theorem we obtained a countable distributionally scrambled invariant set. Another interesting question is how large can this set be.

Theorem 2. *Let (X, d) be a compact metric space, $\#X > 1$, and let $f : X \to X$ be a continuous mapping with WSP* which has a fixed point and infinitely many periodic points with mutually different periods. Then there is a dense invariant distributionally scrambled (of type 1) Mycielski set.*

We say that a continuous map from the unit cube I^k into itself exhibits invariant distributional chaos of type 1 almost everywhere (briefly invariant DC1 a.e.) if there exists a distributionally scrambled set $D \subset I^k$ of type 1 such that $\lambda(D) = 1$ and D is invariant.
If $D \subset I^k$ is a dense union of perfect sets then D is homeomorphic to a set of full Lebesgue measure. An appropriate homeomorphism is obtained by application of the Oxtoby-Ulam theorem.

Corollary 1. *Every map $f \in \mathcal{C}(I^k)$ with WSP*, a fixed point, and infinitely many periodic points with mutually different periods is conjugate to some map $g \in \mathcal{C}(I^k)$ which exhibits invariant DC1 a.e.*

B. **Scrambled and distributionally scrambled n-tuples.** In the interval case, distributional chaos is equivalent to the positivity of topological entropy, which was shown in [9] to be equivalent to the existence of a scrambled triple. Hence distributionally chaotic pairs are always accompanied by scrambled triples. This is no longer true for mappings acting on a general compact metric space. Both of the following counterexamples are shift spaces, their construction is based on the well-known Thue-Morse sequence.

Theorem 3. *There exists a dynamical system with an infinite extremal distributionally 2-scrambled set but without any scrambled triple.*

Theorem 4. *There exists a noncompact dynamical system with an uncountable extremal distributionally 2-scrambled set but without any scrambled triple.*

C. **Distributional chaos and factors.** Semiconjugacy is used as a common tool for proving topological chaos or positive topological entropy. The usual technique is to find a semiconjugacy π with a chaotic system and transfer the chaos to the extension. By continuity of π, the topological entropy of the extension is not smaller than the entropy of the factor system. Unfortunately, semiconjugacy may not automatically guarantee the distributional chaos in the extension.

Theorem 5. *There exists a distributionally chaotic system of type 1 (possessing an uncountable DC1 set) which is semiconjugated to an extension with no distributionally chaotic pair (of type 1 or 2).*

D. **Iteration problem for distributional chaos.** For completeness, we first state all existing results about $DC2\frac{1}{2}$ from [14]. By simple observation, we can see that if $(x, y) \in X^2$ is DC2$\frac{1}{2}$, then it is Li-Yorke scrambled. Indeed, $\Phi^*_{(x,y)}(0) > 0$ implies (x, y) being proximal (for distal pairs, $\Phi^*_{(x,y)}(0) = 0$). Similarly, $\Phi_{(x,y)}(0) < 1$ implies (x, y) being not asymptotic (for asymptotic pairs, $\Phi_{(x,y)}(0) = 1$).
DC2$\frac{1}{2}$ is strictly stronger than DC3 (any distal DC3 system must be without DC2$\frac{1}{2}$ pairs) and strictly weaker than DC2 (see the example of a strictly DC2$\frac{1}{2}$ system in [14]). By results in [7], positive topological entropy implies existence of an uncountable DC2 set, hence strictly DC2$\frac{1}{2}$ systems must have zero topological entropy.
Like DC1 and DC2, DC2$\frac{1}{2}$ is conjugacy invariant. Let f and g be topologically conjugate continuous maps of a compact metric space. Then f is DC2$\frac{1}{2}$ if and only if g is DC2$\frac{1}{2}$.
We claim that DC2$\frac{1}{2}$ is also iteration invariant while DC3 is not. That legitimates the attempt to replace DC3 by its slightly strengthened version denoted by DC2$\frac{1}{2}$.

Theorem 6. *For any integer $N > 1$, the function f^N is distributionally chaotic of type $2\frac{1}{2}$ if and only if f is as well.*

Theorem 7. *Distributional chaos of type 3 is not iteration invariant.*

6. Publication concerning the thesis

(A) J. Doleželová, Distributionally scrambled invariant sets in a compact metric space, Nonlinear Analysis 79 (2013), 80–84. ISSN 0362-546X (Netherlands) (IF 1.3)

(B) J. Doleželová, Scrambled and distributionally scrambled n-tuples, J. Difference Equ. Appl. 20 (2014), 1169–1177. ISSN 1023-6198 (GB) (IF 0.9)

(C) J. Doleželová-Hantáková, Distributional chaos and factors, J. Differ. Equ. Appl. 22 (2016), 99–106. ISSN 1023-6198 (GB) (IF 0.9)

(D) J. Hantáková, Iteration problem for distibutional chaos, submitted to Chaos, Solitons & Fractals in November 2016

7. Other related publication, not used in the thesis

J. Doleželová-Hantáková, S. Roth and Z. Roth, On the weakest version of distributional chaos, Int. J. Bifur. Chaos 26 (2016), no. 14, 1650235. ISSN 0218-1274 (Singapore) (IF 1.0)

8. Quotations by other authors

(1) H. Wang and L. Wang, The weak specification property and distributional chaos, Nonlin. Anal. 91 (2013), 46–50. (ref. A)

(2) M. Foryš, P. Oprocha and P. Wilczyński, Factor maps and invariant distributional chaos, J. Diff. Equ. 256 (2014), 475–502. (ref. A)
(3) L. D. Wang and H. Wang, Distributionally scrambled set and minimal set, Science China Math. 57 (2014), 1953–1960. (ref. A)
(4) H. Bruin and P. Oprocha, On observable Li-Yorke tuples for interval maps, Nonlinearity 28 (2015), 1675–1694. (ref. B)
(5) L. Wang, H. Wang, and G. Huang, Minimal sets and omega-chaos in expansive systems with weak specification property, Discrete Cont. Dynam. Sys 35 (2015), 1231–1238. (ref. A)
(6) J. Li and X. D. Ye, Recent development in chaos theory in topological dynamics, Acta Math. Sinica, English Series, 32 (2015), 83–114. (ref. B)
(7) L. Wang, X. Wang, F. Lei and H. Liu, Asymptotic average shadowing property, almost specification property and distributional chaos, Modern Physics Letters B 30 (2016) Art. No. 1650001. (ref. A)

9. Presentations

(1) 19th European Conference on Iteration Theory 2012 (ECIT 2012), Ponta Delgada, Azores, Portugal, September 9 - 15, 2012. Title of the talk: "Distributionally scrambled invariant sets in compact metric space."
(2) 16th Czech-Slovak Workshop on Discrete Dynamical Systems (CSWDDS 2012), Pustevny, Beskydy mountains, June 11 - 15, 2012. Title of the talk: "Distributionally scrambled invariant sets in compact metric space.
(3) Mathematical competition of students of Czech and Slovak universities – SVOČ, Opava, May 21 - 23, 2013. The first prize in section Mathematical Analysis for the contribution "Solution of two problems on distributionally chaotic dynamical systems."
(4) Visegrad Conference on Dynamical Systems, Olsztyn, Poland, September 2 - 6, 2013. Title of the talk: "Scrambled and distributionally scrambled n-tuples."
(5) 38th Summer Symposium in Real Analysis, Prague, July 7 - 13, 2014. Title of the talk: "Distributional chaos – recent results."
(6) 20th European Conference on Iteration Theory (ECIT 2014), Lagow, Poland, September 14 - 20, 2014. Title of the talk: "Distributional chaos via semiconjugacy."
(7) Progress on Difference Equations, Covilha, Portugal, June 15 - 18, 2015. Title of the talk: "The two versions of the weakest form of distributional chaos."
(8) The Sixth Visegrad Conference, Dynamical Systems, Prague, July 6 - 10, 2015. Title of the talk: "The two versions of the weakest form of distributional chaos."
(9) 40th Summer Symposium in Real Analysis, Sarajevo, Bosnia and Herzegovina, June 19 - 25, 2016. Title of the talk: "Iteration problem for distributional chaos."
(10) 21th European Conference on Iteration Theory (ECIT 2016), Innsbruck, Austria,September 4 - 10, 2016. Title of the talk: "Distributional chaos for iterated function."

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