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Recurrence in systems with randomly perturbed trajectories

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Contents

1. Introduction

The main objects of interest of discrete dynamical systems are autonomous discrete dynamical systems, i.e., pairs of the form (X, f), where X is a compact metric space and f is a continuous selfmap of X. One of possible generalizations of autonomous systems are non-autonomous systems, i.e., pairs of the form $(X, f_{0,\infty})$, where $f_{0,\infty}$ is a sequence of continuous selfmaps of X. We study properties of both, autonomous and non-autonomous, systems with additive stochastic perturbations. Such systems have been introduced by Janková and Smítal in nineties (see [5, 7, 6]). These perturbations are involved, because they often exist in practical situations.

The Thesis consists of three independent papers [1]–[3] (one is published, one is accepted for publication and one is submitted). In the Thesis the definition of recurrence and uniform recurrence for discrete dynamical systems with randomly perturbed trajectories are introduced and sufficient conditions for recurrence are given. The equivalence of recurrence and uniform recurrence for such systems is proved. The definition of nonchaoticity with respect to small random perturbations for a uniformly convergent sequence of selfmaps of the unit interval is introduced and a sufficient condition for this type of nonchaoticity is given.

In the first part of the Thesis discrete dynamical systems generated by a selfmap of the unit interval under additive stochastic perturbations are concerned. For such systems the definition of recurrence is given. Some conditions under which the points of finite and infinite ω-limit sets are recurrent in the sense of this definition are proved.

The second part of the Thesis provides a more general definition of recurrence and analogous results concerning recurrence for points of finite ω-limit sets in higher, finite dimensional cases. It is proved that recurrence implies uniform recurrence for systems with randomly perturbed trajectories. An analogous notion of recurrence for one-dimensional non-autonomous case is also studied.

In the third part of the Thesis some results concerning recurrence for uniformly convergent non-autonomous discrete dynamical systems on the unit interval are presented. In particular, they concern the points of finite and infinite ω-limit sets of the limit function. A sufficient condition for nonchaoticity with respect to small random perturbations of a uniformly convergent sequence of selfmaps of the unit interval is given.

2. Basic terminology and notation

The set of all continuous selfmaps of a compact metric space X is denoted by $\mathcal{C}(X)$. By $\mathbb{N}$ we mean the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $f \in \mathcal{C}(X)$, $f^0(x) = x$ and for $n \in \mathbb{N}_0$, f^n is the *n-th iteration* of f defined by $f^{n+1}(x) = f(f^n(x))$. Let $f \in \mathcal{C}(X)$ and $m \in \mathbb{N}$. A point $x \in X$ is a *fixed point* of f if $f(x) = x$. We denote the set of all fixed points of f by $\mathrm{Fix}(f)$. The fixed point x is *attractive* if there exists a neighborhood U of x such that for each $y \in U$,

$$\lim_{k\to\infty} f^k(y) = x.$$

A point $x \in X$ is a *periodic point* of f with *period* n if $f^n(x) = x$ and $f^k(x) \neq x$ for any $k = 1, \dots, n-1$. We denote the set of all periodic points of f by $\mathrm{Per}(f)$. If x is periodic with period n, then x is *attractive* for f if at least one point of $x, f(x), \dots, f^{n-1}(x)$ is an attractive fixed point for f^n. We say that $A \subseteq X$ is *invariant* for f if $f(A) \subseteq A$. We say

that $f \in \mathcal{C}(I)$ is *of type* 2^∞ if it has a periodic point of period 2^n for all $n \in \mathbb{N}_0$ and no periodic points of other periods. The *ω-limit set* of a point $x \in X$ under $f \in \mathcal{C}(X)$ is the set of all limit points of the sequence $(f^n(x))_{n=0}^\infty$. We say that an ω-limit set $\tilde{\omega}$ is *maximal* if for each ω-limit set $\tilde{\omega}_1$, $\tilde{\omega}_1 \subseteq \tilde{\omega}$ or $\tilde{\omega}_1 \cap \tilde{\omega} = \emptyset$.
Recall the definition of recurrence and uniform recurrence for standard (autonomous) discrete dynamical systems (with no perturbations). Let $f \in \mathcal{C}(X)$. We call $x \in X$ *recurrent* if for each open neighborhood U of x there is a strictly increasing sequence of positive integers (n_k) such that $f^{n_k}(x) \in U$. We call x *uniformly recurrent* if for each neighborhood U of x there is $K \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $f^m(x) \in U$ implies $f^{m+i}(x) \in U$ for some $i \in \{1, \ldots, K\}$.

By $f_{0,\infty}$ we mean a sequence of selfmaps $(f_0, f_1, \ldots)$ of a compact metric space X. The sequence arising from $f_{0,\infty}$ by removing the first k elements is denoted by $f_{k,\infty}$. Let $f_{0,\infty}$ be a sequence in $\mathcal{C}(X)$. By the trajectory of $x_0 \in X$ under $f_{0,\infty}$ we mean the sequence $(x_n)_{n=0}^\infty$ defined by $x_{n+1} = f_n(x_n)$ for each $n \in \mathbb{N}_0$. A discrete (autonomous) dynamical system (X, f), where $f \in \mathcal{C}(X)$, can be obtained as a special case of the non-autonomous system $(X, f_{0,\infty})$ by taking $f_{0,\infty} = (f, f, \ldots)$. We are particularly interested in non-autonomous systems $(X, f_{0,\infty})$, where $f_{0,\infty}$ converges uniformly to the limit function f. Such systems are called uniformly convergent. There are several properties concerning, e.g., relations between chaoticity or ω-limit sets of a uniformly convergent non-autonomous system and the autonomous system generated by the limit function (see [9]).

Whenever we consider systems with randomly perturbed trajectories, we work with random variables defined on a fixed probability space (Ω, Σ, P). By I^m we denote the Cartesian product of m intervals $I = [0, 1]$, where $m \in \mathbb{N}$. Since we work with random perturbations, for each $f \in \mathcal{C}(I^m)$ we consider a continuous extension of f to $\mathbb{R}^m$ such that $f(\mathbb{R}^m \setminus I^m) \subseteq f(\partial I^m)$, where ∂A means the boundary of A. This extension will be denoted by f as well. Let $\delta > 0$ and let $f_{0,\infty}$ be a sequence in $\mathcal{C}(I^m)$. For $x \in \mathbb{R}^m$ we use the notation $x = (x^{(1)}, \ldots, x^{(m)})$. An *$(f_{0,\infty}, \delta)$-process that begins at $x_0 \in I^m$* is a sequence of random variables defined by the formula $X_{n+1} = f_n(X_n) + (\xi_n^{(1)}, \ldots, \xi_n^{(m)})$, $n \in \mathbb{N}_0$ and $X_0 = x_0$, where all $\xi_k^{(j)}$, $j = 1, \ldots, m$, $k = 0, 1, \ldots$ are independent and have uniform continuous distributions on $[-\delta, \delta]$. Then a point $x \in I^m$ is called *$(f_{0,\infty}, \delta)$-recurrent* if for each open neighbourhood U of x and each $\delta' \in (0, \delta)$,

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k \in U\}\right) = 1, \tag{1}$$

where (X_n) is any $(f_{0,\infty}, \delta')$-process that begins at x. If (1) holds and there are $K \in \mathbb{N}_0$ and $\lambda \in (0, 1]$ such that for each $n \in \mathbb{N}_0$,

$$P\left(\bigcup_{i=nK+1}^{(n+1)K} \{X_i \in U\}\right) \geq \lambda,$$

we call x *uniformly (f, δ)-recurrent*. By taking

$$f_{0,\infty} = (f, f, \ldots)$$

we obtain the definitions of an (autonomous) (f, δ)-process and an (f, δ)-recurrent point.

By $\|\cdot\|$ we mean the maximum norm defined on $\mathbb{R}^m$, i.e.,

$$\|x\| = \max_{i=1,\ldots,m} |x^{(i)}|$$

for $x \in \mathbb{R}^m$. The open ball of radius $r > 0$ centred at x is denoted by $B(x, r)$.
The following definitions are related to chaotic properties of continuous selfmaps of the unit interval. We say that $f \in \mathcal{C}(I)$ is *nonchaotic* if for each $x \in I$ and each $\varepsilon > 0$ there exists $p \in \mathrm{Per}(f)$ such that $\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \varepsilon$. Notice that for each $f \in \mathcal{C}(I)$, f is either nonchaotic or it is chaotic in the sense of Li and Yorke (i.e., there exists an uncountable set $S \subseteq I$ such that for all $x, y \in S$ and $x \neq y$, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$). We call f *nonchaotic stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $g \in \mathcal{C}(I)$ and each $x \in I$, $\|f - g\| < \delta$ implies $\limsup_{n\to\infty} |g^n(x) - g^n(p)| < \varepsilon$ for some $p \in \mathrm{Per}(g)$. We call f *nonchaotic with respect to small random perturbations* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\delta' \in (0, \delta)$ and each $x_0 \in I$,

$$P\left(\exists p \in \mathrm{Per}(f)\colon \ \limsup_{n\to\infty} |X_n - f^n(p)| < \varepsilon\right) = 1, \tag{2}$$

where (X_n) is any (f, δ')-process which begins at x_0. We generalize the definition of nonchaoticity with respect to small random perturbations for a uniformly convergent sequence of continuous selfmaps of the interval: a sequence $f_{0,\infty}$ in $\mathcal{C}(I)$, which converges uniformly to $f \in \mathcal{C}(I)$, is *nonchaotic with respect to small random perturbations* if for each $\varepsilon > 0$ there exist $K \in \mathbb{N}$ and $\delta > 0$ such that for each integer $k > K$, for each $\delta' \in (0, \delta)$ and for each $x_0 \in I$, (2) holds, where (X_n) is any $(f_{k,\infty}, \delta')$-process which begins at x_0.

3. Main results of the Thesis

The results of the Thesis can be divided into several groups. The first group concerns two types of recurrence defined for discrete dynamical systems with randomly perturbed trajectories. The following Theorem provides a condition equivalent to the definition of (f, δ)-recurrence.

Theorem 1 (see [3])**.** *Let $f \in \mathcal{C}(I^m)$, $m \in \mathbb{N}$. Let (X_n) be an (f, δ)-process that begins at $x_0 \in I^m$. Then x_0 is (f, δ)-recurrent if and only if for each neighborhood U of x_0, $P(\bigcup_{k=1}^{\infty}\{X_k \in U\}) = 1$.*

The following result concerns some properties of periodic points. It is formulated in the most general form for a non-autonomous discrete dynamical system on the m-dimensional cube with randomly perturbed trajectories (see [2]) and it provides a class of (f, δ)- and $(f_{0,\infty}, \delta)$-recurrent points. In [1] it was shown in dimension one for autonomous systems and then for finite dimensional autonomous case in [3].

Theorem 2 (see [2])**.** *Let $f_{0,\infty}$ be a sequence in $\mathcal{C}(I^m)$, $m \in \mathbb{N}$. Assume that $f_{0,\infty}$ converges uniformly to a function $f \in \mathcal{C}(I^m)$. Let $x_0 \in I^m$ be an attractive periodic point of f with period n. Then there exist $K \in \mathbb{N}$ and $\delta > 0$ such that for each integer $k > K$, x_0 is $(f_{k,\infty}, \delta)$-recurrent.*

The following theorem concerns some properties of infinite ω-limit sets and it gives another class of (f, δ)- and $(f_{0,\infty}, \delta)$-recurrent points. It is formulated in its full generality, i.e., for non-autonomous systems on the unit interval. The domain must be I since the proof is based on the fact that continuous selfmaps of the unit interval of type 2^∞ have infinite ω-limit sets of a special structure (see [4] and [8]). Notice that this result is also true for autonomous discrete dynamical systems, since they are a special case of non-autonomous systems (see Theorem 4 in [1]), but in the formulation of Theorem 4 in [1] some assumptions were overlooked. Some examples showing the necessity of these assumptions can be found in [1] and [2].

Theorem 3 (see [2])**.** *Let $f \in \mathcal{C}(I)$ be of type 2^∞. Assume that f has a maximal infinite ω-limit set $\tilde{\omega}$. Assume that there exists $m \in \mathbb{N}$ such that the minimal closed and invariant interval V containing $\tilde{\omega}$ contains exactly m different maximal ω-limit sets, m periodic orbits of period 2^n for each $n \in \mathbb{N}$ and none of them is attractive. Let $f_{0,\infty}$ be a sequence in $\mathcal{C}(I)$, which converges uniformly to f. Then there exist $K \in \mathbb{N}$ and $\delta > 0$ such that for each integer $k > K$, every point $x \in \tilde{\omega}$ is $(f_{k,\infty}, \delta)$-recurrent.*

It is known, that for standard discrete dynamical systems (with no perturbations) uniform recurrence implies recurrence, but the opposite implication is not true. Similar notions related to processes with randomly perturbed trajectories are equivalent.

Theorem 4 (see [3])**.** *Let $x_0 \in I^m$ be (f, δ)-recurrent with $\delta > 0$, $f \in \mathcal{C}(I^m)$, $m \in \mathbb{N}$. Then x_0 is uniformly (f, δ)-recurrent.*

The next group concerns non-chaoticity of discrete dynamical systems. We give a generalization of the result from [5] which states, that $f \in \mathcal{C}(I)$ is nonchaotic with respect to small random perturbations if f is nonchaotic stable.

Theorem 5 (see [2])**.** *Let $f_{0,\infty}$ be a sequence in $\mathcal{C}(I)$ converging uniformly to $f \in \mathcal{C}(I)$. Let f be nonchaotic stable. Then $f_{0,\infty}$ is nonchaotic with respect to small random perturbations.*

4. Conferences

- [C1] Czech-Slovak-Spanish Workshop on Discrete Dynamical Systems, La Manga del Mar Menor, Spain, September 20–24, 2010.
- [C2] Visegrad Conference on Dynamical Systems, Banská Bystrica, Slovakia, June 27 – July 3, 2011.
  Talk: "Recurrence in systems with random perturbations."
- [C3] Visegrad Conference on Dynamical Systems, Olsztyn, Poland, September 2–6, 2013.
  Talk: "Recurrence in systems with random perturbations in the finite dimensional case."
- [C4] Conference on Ulam's Type Stability, Rytro, Poland, June 2–6, 2014.
  Talk: "Chaotic behavior of discrete dynamical systems with randomly perturbed trajectories."
- [C5] The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Madrid, Spain, July 07–11, 2014.
  Talk: "Chaotic behavior of non-autonomous systems with randomly perturbed trajectories."
- [C6] The 18th Czech-Slovak Workshop on Discrete Dynamical Systems, Malenovice, Czech Republic, September 8–12, 2014.
  Talk: "Recurrence in uniformly convergent non-autonomous systems with randomly perturbed trajectories."

Publications constituting the Thesis

[1] L. Szała. Recurrence in systems with random perturbations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 23(6):1350110, 2013.

[2] L. Szała. Chaotic behaviour of uniformly convergent nonautonomous systems with randomly perturbed trajectories. *arXiv:1408.2569*, 2014. Submitted to J. Differ. Equ. Appl.

[3] L. Szała. Recurrence in systems with randomly perturbed trajectories on the n-dimensional cube. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 24(11):1450137, 2014.

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