REPORT ON THE HABILITATION THESIS OF KAREL HASÍK.

The nice criterion of Kuang and Freedman allows to determine the uniqueness of the limit cycle of a class of systems of Gause-Kolmogorov type that appear frequently in models of populations dynamics. Its statement is a follows: consider system

$$\begin{aligned} \dot{x} = & \; xg(x) - yp(x), \\ \dot{y} = & \; y(q(x) - \gamma), \end{aligned} \tag{1}$$

in the first quadrant, with $\gamma > 0$. Assume that there exist $K > 0$ and $x^* > 0$ such that:

(i) $p(0) = 0$, $p'(x) > 0$, for $0 < x < K$,
(ii) $g(K) = 0$, $(x - K)g(x) < 0$, for $0 < x \neq K$,
(iii) $q(x^*) = \gamma$, $(q(x) - \gamma)(x - x^*) > 0$, for $x \in [0, K] \setminus \{x^*\}$,
(iv) the unique equilibrium point (x^*, y^*) in the region $R = \{(x, y) : x \in [0, K], y > 0\}$ is unstable,
(v) for $0 < x < K$,

$$\Phi(x) := \frac{d}{dx} \left(\frac{xg'(x) + g(x) - xg(x)p'(x)/p(x)}{-\gamma + q(x)} \right) \leq 0, \quad x \neq x^*.$$

Then system (1) has exactly one periodic orbit which is a globally asymptotically stable limit cycle.

In fact, the stability of the critical point is given by the so-called Rosenzweig and McArthur criterion. It says that that the (hyperbolic) instability of (x^*, y^*) is equivalent to

$$\frac{d}{dx} \left(\frac{xg(x)}{p(x)} \right) \bigg|_{x=x^*} > 0.$$

Although the set of the above hypotheses looks very restrictive it holds that they are natural and applicable in several concrete models.

It is worth to mention that when the above hypotheses are not satisfied there can be systems having more limit cycles. Hence to get weaker hypotheses under which these systems have uniqueness of limit cycles it is not easy at all and very relevant for concrete applications. This is precisely the goal of the first three papers [H1,H2] and [H3] presented in this Habilitation thesis. More concretely, when $q(x) = cp(x)$, in the first work the intervals where the inequalities are satisfied is diminished, giving as a consequence a wider set of applications. In [H2] a kind of weight function $W(x)$ is introduced to increase once more the models where uniqueness can be proved. In [H3] the author studies the properties of useful W

and characterizes their existence in terms of inequalities between the functions appearing in the model.

The tools used in these three publications are standard in the qualitative theory of planar autonomous ordinary differential equations. The author studies the stability of the limit cycle thought the integral of the divergence and proves that it has a given sign. By the classical Poincaré-Bendixson criterion, and due to the uniqueness of the critical point in the first quadrant, the uniqueness of the limit cycle follows. The study of this integral is difficult and uses many tricks and inequalities based of the symmetries of the problem. The results of the three papers are illustrated with many examples.

It would have been interesting if the author would also have addressed to the following questions:

- Is it possible to obtain results of non-uniqueness of the limit cycles using similar tools?
- Is there some relation between the approach of the papers and other approaches to the study of the number of limit cycles in the plane, as for instance the use of Dulac functions?

The second part of the Thesis deals with a different problem that has also biological motivations.

Mathematical modeling of dynamical processes in a great variety of natural phenomena leads in general to non-linear partial differential equations. There is a particular class of solutions for these non-linear equations that are of considerable interest. They are the traveling wave solutions (TWS). Such waves are special solutions which do not change their shape and which propagate at constant speed, c. The profiles of these special solutions are the result of a balance between nonlinearity and dissipation. They appear in many different models and they usually write as

$$u(t,x) = \phi(v \cdot x + ct), \quad |v| = 1.$$

One of the most famous equations is the celebrated Fisher-Kolmogorov reaction-diffusion partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)\,(1-u(t,x)),$$

introduced in 1937 in two classical papers to model the spreading of biological populations. It is well known that for this equation, TWS with the limit behavior

$$\lim_{s\to-\infty} \phi(s) = 0, \quad \lim_{s\to+\infty} \phi(s) = 1, \tag{2}$$

only exist for $c \geq 2$. These type of TWS are also called wavefront (or traveling front).

The main goal of the last two papers of the Memoire, [HT1] and [HT2], is to the study the existence of wavefronts and a generalization of them, called semi-wavefronts, for the m-dimensional version of the above equation with some delay τ:

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\,(1 - u(t-\tau,x)). \qquad (3)$$

For these waves the limit conditions given in (2) are relaxed.

Paper [HT2] constitutes the more technical and difficult part of the Habilitation thesis. Using several known techniques the authors reduced the existence of semi-wave fronts to the existence of a fixed point of an integral operator, $\mathcal{A}$. The more difficult part is the proof of the existence of a fixed point in this setting. They succeed in posing the problem in a framework where Shauder fixed point applies for a modified version of the operator $\mathcal{A}$.

The results obtained are difficult and relevant and show three different possibilities for the right behavior of the semi-wave front, one of them being of oscillatory type. The results are proved by a clever interlace of several techniques going from functional analysis, sub and supper solutions of delayed ODE and several methods of dynamical systems like the use of the Schwarzian derivative.

In [HT2] it is shown an interesting relation between existence of TWS for (3), for some delay τ, and a famous conjecture due to Wright. Recall that this conjecture says that the solution $\psi = 1$ of the delayed ODE

$$\psi'(t) = \psi(t)\,(1 - \psi'(t-\tau)),$$

is globally stable in the domain of positive solutions $\psi > 0$, if and only if $\tau \leq \pi/2$. This stability is known to be true when $\tau \leq 3/2$. In the paper [HT2] it is proved that when $c \geq 2$ and $\tau \in [0, 3/2]$ the delayed Fisher-Kolmogorov equation has at least one positive TWS of front type.

As a resume I can say that the results presented in this Habilitation thesis are non-trivial and relevant and also have a good level, comparable to the medium publications on dynamical systems and analysis. The journals where the publications appear are top journals. I rate publications [H1],[H2] and [H3] as good and publications [HT1] and [HT2] as excellent. Shortly, my report for the habilitation of Karel Hasík is very positive.

Bellaterra, June 2015, 18

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