Silesian University in Opava
Mathematical Institute

Zdeněk Kočan

Triangular maps of the square

Abstract of the Ph.D. Thesis
July 2002

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav

Zdeněk Kočan

Trojúhelníková zobrazení čtverce

Autoreferát dizertační práce
Červenec 2002

Matematická analýza

Výsledky tvořící dizertační práci byly získány během doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 1996–2002.

Dizertant:	RNDr. Zdeněk Kočan Matematický ústav SU, Opava
Školitel:	Prof. RNDr. Jaroslav Smítal, DrSc. Matematický ústav SU, Opava
Školící pracoviště:	Matematický ústav SU, Opava
Oponenti:	Prof. RNDr. František Neuman, DrSc. Matematický ústav AV ČR, Brno
	Doc. RNDr. Katarína Janková, Ph.D. Fakulta matematiky, fyziky a informatiky UK, Bratislava

Autoreferát byl rozeslán: 1. 10. 2002

Státní doktorská zkouška a obhajoba dizertační práce se konají dne 24. 10. 2002 v 9:30 hod., před oborovou radou doktorského studia matematické analýzy v zasedací místnosti rektorátu Slezské univerzity, Bezručovo nám. 13, Opava.

S dizertací je možno se seznámit v knihovně Matematického ústavu SU, Bezručovo nám. 13, Opava.

Prof. RNDr. Jaroslav Smítal, DrSc.
Matematický ústav SU, Opava
Předseda oborové rady

Contents

1. Introduction

The thesis is based on two papers [Ko1], [Ko2]. The common subject is the theory of discrete dynamical systems generated by triangular maps of the unit square into itself.

All parts give relations between properties of triangular maps of the square (we include also some known results). These properties are mutually equivalent in the case of continuous maps of a compact interval into itself. We look for equivalent ones with the property "the map has zero topological entropy". The problem was originally formulated by Sharkovsky in 1989. The first part (Section 3) of the thesis studies the properties of a special case of triangular maps that are non-decreasing on the fibres. The second part (Section 4) provides relations between some properties of triangular maps of the square. The third part (Section 5) studies relations between thirteen properties of triangular maps that are mutually equivalent whenever the fibre maps are non-decreasing. Even in this case there are some open problems.

2. Basic terminology and notation

Throughout this abstract, $I = [0, 1]$ is the unit compact interval, I^2 the unit square, and X a compact metric space with a metric ρ. Let $\mathcal{C}(X, X)$ be the set of continuous mappings of X into itself, $\mathbb{N}$ the set of positive integers, and $\mathbb{N}_0$ the set of non-negative integers. For $\varphi \in \mathcal{C}(X, X)$, let $\varphi^n(x)$ denote the n-th iterate of φ at x, for $n \in \mathbb{N}$ and $x \in X$. The set of cluster points of the sequence $(\varphi^n(x))_{n\in\mathbb{N}}$ is the *ω-limit set $\omega_\varphi(x)$ of x*. Let $\pi : I^2 \to I$ be the projection $(x, y) \mapsto x$.

Let $f : I \to I$, and $g_x : \{x\} \times I \to I$, for $x \in I$. A map $F \in \mathcal{C}(I^2, I^2)$ such that $F(x, y) = (f(x), g_x(y))$, for any x, y in I, is a *triangular map*, f is the *base of* F, and the set $I_x := \{x\} \times I$ is the *layer over* x. Throughout the paper, F always denotes a triangular map, and f its base.

We proceed by the list of properties of continuous maps of a compact metric space into itself; the symbols used in them are explained below.

(P1) $h(\varphi) = 0$
(P2) $h(\varphi|\ \mathrm{CR}(\varphi)) = 0$
(P3) $h(\varphi|\ \Omega(\varphi)) = 0$
(P4) $h(\varphi|\ \omega(\varphi)) = 0$
(P5) $h(\varphi|\ \mathrm{C}(\varphi)) = 0$

The research was supported, in part, by the Grant Agency of Czech Republic, grant No. 201/97/001, and No. 201/00/0859, and by the Czech Ministry of Education, project MSM 192400002. Support of these institutions is gratefully acknowledged.

(P6) $h(\varphi|\ \mathrm{Rec}(\varphi)) = 0$
(P7) $h(\varphi|\ \mathrm{UR}(\varphi)) = 0$
(P8) $h(\varphi|\ \mathrm{AP}(\varphi)) = 0$
(P9) $h(\varphi|\ \mathrm{APB}(\varphi)) = 0$
(P10) $h(\varphi|\ \mathrm{Per}(\varphi)) = 0$
(P11) Every cycle is simple
(P12) Period of any cycle is a power of 2
(P13) There is no minimal set with positive topological entropy
(P14) φ has no homoclinic trajectory
(P15) $\varphi|\ \mathrm{CR}(\varphi)$ is non-chaotic
(P16) $\varphi|\ \Omega(\varphi)$ is non-chaotic
(P17) $\varphi|\ \omega(\varphi)$ is non-chaotic
(P18) $\varphi|\ \mathrm{C}(\varphi)$ is non-chaotic
(P19) $\varphi|\ \mathrm{Rec}(\varphi)$ is non-chaotic
(P20) $\varphi|\ \mathrm{UR}(\varphi)$ is non-chaotic
(P21) $\mathrm{UR}(\varphi) = \mathrm{Rec}(\varphi)$
(P22) Every ω-limit set contains a unique minimal set
(P23) No infinite ω-limit set contains a cycle
(P24) Every ω-limit set either is a cycle or contains no cycle

In the sequel, $\mathrm{CR}(\varphi)$ denotes the set of *chain recurrent points* of φ. Thus, $x \in \mathrm{CR}(\varphi)$ if, for any $\varepsilon > 0$, there is a sequence of points $(x_i)_{i=0}^n$ with $x_0 = x$ and $x_n = x$ such that $\rho(x_{i+1}, \varphi(x_i)) < \varepsilon$, for $i = 0, 1, 2, \ldots, n-1$. $\Omega(\varphi)$ is the set of *non-wandering points* of φ, i.e., $x \in \Omega(\varphi)$ if, for any neighbourhood U of x, there is an $n \in \mathbb{N}$ with $\varphi^n(U) \cap U \neq \emptyset$. By $\omega(\varphi)$ we denote the set of *ω-limit points* of φ, and by $\mathrm{Rec}(\varphi)$ the set of *recurrent points* of φ, i.e., the set of $x \in X$ such that $x \in \omega_\varphi(x)$, while $\mathrm{C}(\varphi) = \mathrm{cl}(\mathrm{Rec}(\varphi))$ is the *centre* of φ. $\mathrm{UR}(\varphi)$ denotes the set of *uniformly recurrent points* of φ, i.e., the set of $x \in X$ such that for any neighbourhood U of x, there is an $n \in \mathbb{N}$ such that if $\varphi^m(x) \in U$, where $m \geq 0$, then $\varphi^{m+k}(x) \in U$ for some k with $0 < k \leq n$. By $\mathrm{AP}(\varphi)$ we denote the set of *almost periodic points* of φ, i.e., the set of $x \in X$ such that for any neighbourhood U of x, there is an $n \in \mathbb{N}$ such that $\varphi^{in}(x) \in U$, for any i. $\mathrm{APB}(\varphi)$ represents the set of *almost periodic points of φ in the sense of Bohr*; thus $x \in \mathrm{APB}(\varphi)$ if for any neighbourhood U of x, there is a $k \in \mathbb{N}$ such that for any i there is a j with $i < j \leq i + k$ and $\varphi^j(x) \in U$. $\mathrm{Per}(\varphi)$ is the set of *periodic points* of φ.

Denote by $h_\rho(\varphi|\ M)$ the *topological entropy of the map φ with respect to the compact subset M* and by $h_\rho(\varphi)$ the *topological entropy of the map φ*. If no confusion can arise we write h instead of h_ρ.

Let $\varphi \in \mathcal{C}(I,I)$ and let $\alpha = \{x_1, x_2, \dots, x_{2^n}\} \subset I$, where $n \in \mathbb{N}_0$, be a cycle of φ with period 2^n such that $x_1 < x_2 < \ldots < x_{2^n}$. Then α is a *simple cycle of* φ, if either $n = 0$ (and $\alpha = \{x\}$ is a fixed point), or $n > 0$ and the sets $\{x_1, x_2, \dots, x_{2^{n-1}}\}$, $\{x_{2^{n-1}+1}, \dots, x_{2^n}\}$ are invariant sets with respect to φ^2, and each of them is a simple cycle of φ^2.

Let α be a cycle of a triangular map F with period 2^k, $k \in \mathbb{N}_0$, such that $\pi(\alpha)$ is a simple cycle of the base f with period $2^n = m$, for some $n \leq k$. Then α is a *simple cycle of* F if, for every $x \in \pi(\alpha)$ and every $z \in \alpha \cap I_x$, $\{F^{im}(z) | \, i = 1, 2, \dots, 2^{k-n}\} \subset I_x$ is a simple cycle of $F^m | \, I_x$ (which is a one-dimensional map $I_x \to I_x$).

A subset M of X is a *minimal set* if $M = \omega_\varphi(x)$, for any $x \in M$.

Let $x \in X$ be a fixed point of φ. A sequence $(x_n)_{n=1}^\infty$ of distinct points in X such that $\varphi(x_{n+1}) = x_n$, for every $n \in \mathbb{N}$, $\varphi(x_1) = x$, and $\lim_{n\to\infty} x_n = x$, is a *homoclinic trajectory related to the point* x. A sequence $(y_n)_{n=1}^\infty$ of distinct points in X such that $\varphi(y_{n+1}) = y_n$, for every $n \in \mathbb{N}$, $\varphi(y_1) = y_k$, for some $k \in \mathbb{N}$ (i.e., $\{y_1, \dots, y_k\}$ is a cycle of period k), and $\lim_{n\to\infty} y_{kn+i} = y_i$ for $i = 1, 2, \dots, k$, is a *homoclinic trajectory related to the cycle* $\{y_1, \dots, y_k\}$.

A map φ is *chaotic* (in the sense of Li and Yorke) if there is a *φ-chaotic pair* $\{x, y\} \subset X$, i.e., points $x, y \in X$ such that

$$0 = \liminf_{n\to\infty} \rho\left(\varphi^n(x), \varphi^n(y)\right) < \limsup_{n\to\infty} \rho\left(\varphi^n(x), \varphi^n(y)\right).$$

For more terminology see standard books like [BC] or [SKSF].

3. Triangular maps non-decreasing on the fibres

In the case of general triangular maps of the square there are many open problems. We consider the special maps which are non-decreasing on the fibres, and try to find relations between (P1)–(P24) in this different situation. The known relations between the properties in this case can be given by the following theorem.

Theorem A. *Consider properties* (P1)–(P24) *of triangular maps non-decreasing on the fibres listed in Section 2. The relations between them are displayed by the graph on Figure 1 where a missing arrow means that there is no implication, except for implications that follow by transitivity. Mutually equivalent properties are situated in one circle. The two open problems are indicated by arrows with query.*

Even in this special case there are two open problems. We conjecture that there is a triangular map F non-decreasing on the fibres such that $\mathrm{UR}(F) = \mathrm{Rec}(F)$ and $F| \, \mathrm{Rec}(F)$ is chaotic, and hence, that the two implications (P21) $\Rightarrow$ (P19) and (P21) $\Rightarrow$ (P20) are invalid.

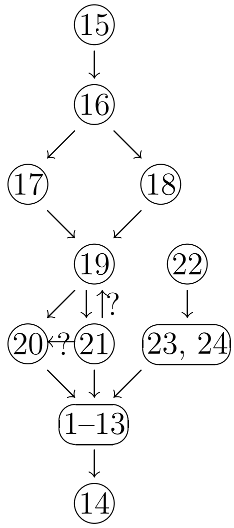

FIGURE 1

4. General triangular maps

The properties (P1), (P10), (P11), (P12), (P14), (P22), (P23), and (P24) are studied in this section. The known results concerning these properties which has been published in [Ko1] can be summarized in the following theorem.

Theorem B. *Consider properties* (P1), (P10), (P11), (P12), (P14), (P22), (P23), *and* (P24) *of triangular maps. The relations between them are displayed by the graph on Figure 2 where a missing arrow means that there is no implication, except for implications that follow by transitivity. Mutually equivalent properties are situated in one circle. The two remaining open problems are indicated by arrows with query.*

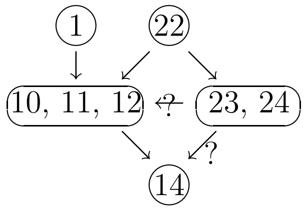

FIGURE 2

5. Properties (P1)–(P13) in the general case

From the previous section it follows that properties (P1)–(P13), which are mutually equivalent in the case of triangular maps non-decreasing on the fibres, are not mutually equivalent in the case of general triangular maps of the square. Their classification seems to be difficult. In this section we only show what is known and what are the open problems.

It is already known that (P1) $\Rightarrow$ (P2) $\Rightarrow$ (P3) $\Rightarrow$ (P4) $\Rightarrow$ (P5) $\Rightarrow$ (P6) $\Rightarrow$ (P7) $\Rightarrow$ (P8) $\Rightarrow$ (P9) $\Rightarrow$ (P10) (this easily follows from the definitions), and (P10) $\Leftrightarrow$ (P11) $\Leftrightarrow$ (P12).

5.1 Lemma ($1 \Leftrightarrow 3 \Leftrightarrow 5$) [BC] *Let (X, ρ) be a compact metric space and $\varphi \in C(X, X)$. Then $h(\varphi) = h(\varphi|\ \Omega(\varphi)) = h(\varphi|\ \mathrm{C}(\varphi))$.*

The following lemma is given in [FSS], for continuous mappings of the interval. Here we rewrite its proof for continuous maps of any compact metric space.

5.2 Lemma *If $A \subset X$ is an invariant set then $h(\varphi|\ A) = h(\varphi|\ \mathrm{cl}(A))$.*

Proof. For simplicity put $A = A_1$ and $\mathrm{cl}(A) = A_2$. For a given $\varepsilon > 0$, $i \in \{1, 2\}$ and a positive integer n, let $s_i(\varphi, \varepsilon, n) \subset A_i$ be a maximal set such that for any $x, y \in s_i(\varphi, \varepsilon, n)$, $x \neq y$, there is an integer k with $0 \leq k < n$ and $|\varphi^k(x) - \varphi^k(y)| > \varepsilon$. From the definition of topological entropy it follows that it suffices to show that for any $\varepsilon > 0$,

$$\#s_1(\varphi, \varepsilon/3, n) \geq \#s_2(\varphi, \varepsilon, n) \tag{1}$$

since the inequality $h(\varphi|\ A_1) \leq h(\varphi|\ A_2)$ is trivial. Thus let $x_1, \ldots, x_m$ be the elements of $s_2(\varphi, \varepsilon, n)$. Since $\mathrm{cl}(A_1) = A_2$, by the continuity of φ there are points $y_1, \ldots, y_m \in A_1$ such that for any $j \in \{1, \ldots, m\}$ and any $r \in \{0, \ldots, n\}$, $|\varphi^r(x_j) - \varphi^r(y_j)| < \varepsilon/3$. Now if for some $k \in \{0, \ldots, n-1\}$ we have $|\varphi^k(x_i) - \varphi^k(x_j)| > \varepsilon$, then clearly $|\varphi^k(y_i) - \varphi^k(y_j)| > \varepsilon/3$. Thus $\{y_1, \ldots, y_m\}$ is contained in a maximal set $s_1(\varphi, \varepsilon/3, n)$ and (1) follows. Q.E.D.

5.3 Corollary ($5 \Leftrightarrow 6$) *$h(\varphi|\ \mathrm{C}(\varphi)) = 0$ if and only if $h(\varphi|\ \mathrm{Rec}(\varphi)) = 0$.*

Proof. The assertion follows the definition of the centre. Q.E.D.

5.4 Lemma ($7 \not\Rightarrow 6$) *There is a triangular map F such that $0 = h(F|\ \mathrm{UR}(F)) < h(F|\ \mathrm{Rec}(F))$.*

Proof. The triangular map F constructed in the proof of Theorem 9 in [K] has the required properties. Q.E.D.

Consider a continuous map $\varphi : X \to X$ of a compact metric space. Let β be the Borel sets of X. Denote by $\mathcal{M}(X, \varphi)$ and $\mathcal{E}(X, \varphi)$ the set of invariant and ergodic measures of φ, respectively.

5.5 Lemma [W, Theorem 6.1] *Let $B \in \beta$, and $\mu \in \mathcal{M}(X, \varphi)$. For any $\varepsilon > 0$, there are C_ε closed and U_ε open such that $C_\varepsilon \subset B \subset U_\varepsilon$ and $\mu(U_\varepsilon \setminus C_\varepsilon) < \varepsilon$.*

5.6 Lemma (7 $\Leftrightarrow$ 13) *$h(F|\ \mathrm{UR}(F)) = 0$ if and only if there is no minimal set with positive topological entropy.*

Proof. The proof is due to José S. Cánovas (cf. [C]).

Consider the sets $\mathrm{UR}(F)$ and $\mathrm{cl}(\mathrm{UR}(F))$. By Lemma 5.5, for any $\varepsilon > 0$, we have

$$C_\varepsilon \subset \mathrm{UR}(F) \subset U_\varepsilon, \quad \text{and} \quad \tilde{C}_\varepsilon \subset \mathrm{cl}(\mathrm{UR}(F)) \subset \tilde{U}_\varepsilon.$$

Then

$$C_\varepsilon \subset \mathrm{UR}(F) \subset \mathrm{cl}(\mathrm{UR}(F)) \subset U_\varepsilon$$

and

$$\begin{aligned} \mu(\tilde{U}_\varepsilon \setminus C_\varepsilon) &\leq \mu((U_\varepsilon \setminus C_\varepsilon) \cup (\tilde{U}_\varepsilon \setminus \tilde{C}_\varepsilon)) \\ &\leq \mu(U_\varepsilon \setminus C_\varepsilon) + \mu(\tilde{U}_\varepsilon \setminus \tilde{C}_\varepsilon) < 2\varepsilon. \end{aligned}$$

So, for any $B \in \beta$ such that $B \subset \mathrm{cl}(\mathrm{UR}(F)) \setminus \mathrm{UR}(F)$ we have $\mu(B) = 0$, which proves that any invariant measure $\mu \in \mathcal{M}(\mathrm{cl}(\mathrm{UR}(F)), F)$ is supported on $\mathrm{UR}(F)$. Hence

$$\begin{aligned} h(F|\ \mathrm{UR}(F)) &= h(F|\ \mathrm{cl}(\mathrm{UR}(F))) \\ &= \sup\{h_\mu(F)|\ \mu \in \mathcal{E}(\mathrm{cl}(\mathrm{UR}(F)), F)\} \\ &= \sup\{h_\mu(F)|\ \mu \in \mathcal{E}(\mathrm{UR}(F), F)\}, \end{aligned}$$

by the variational principle for topological entropy [W, Corollary 8.6.1]. Since for any invariant measure μ it holds that $h_\mu(F) \leq h(F)$ (see [W, Theorem 8.6]).

If for any minimal set M we have that $h(F|\ M) = 0$, then for any ergodic measure in $\mathcal{E}(\mathrm{cl}(\mathrm{UR}(F)), F)$ we have $h_\mu(F) \leq h(F|\ M) = 0$, for some minimal set M. Hence

$$\begin{aligned} h(F|\ \mathrm{UR}(F)) &= \sup\{h_\mu(F)|\ \mu \in \mathcal{E}(\mathrm{UR}(F), F)\} \\ &= \sup\{h(F|\ M)|\ M \text{ is minimal}\} = 0. \end{aligned}$$

The other implication follows from the fact that every minimal set is a subset of $\mathrm{UR}(F)$. Q.E.D.

The known facts, the relations between the properties (P1)–(P13) in the general case, and the open problems are displayed by the graph on Figure 3 where a missing arrow means that there is no implication, except for implications that follow by transitivity. Mutually equivalent properties are situated in one circle. The open problems are indicated by arrows with query.

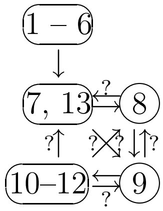

FIGURE 3

6. Publications concerning the thesis

[1] Z. Kočan, *The problem of classification of triangular maps with zero topological entropy*, Annales Mathematicae Silesianae **13** (1999), 181–192.

[2] Z. Kočan, *Triangular maps non-decreasing on the fibres*, Bull. Austral. Math. Soc. (submitted).

7. Quotations by other authors

[3] F. Balibrea, J. L. G. Guirao, J. I. M. Casado, *Description of ω-limit sets of a triangular map on I^2*, Far East J. Dyn. Syst. **3** (2001), no. 1, 87–101.

8. Presentations

[4] 1st Czech-Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, May 31–June 4, 1997. Talk on: "Triangular maps of the square."

[5] 26th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January 23–29, 1998. Talk on: "The problem of classification of triangular maps with zero topological entropy."

[6] 2nd Czech-Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, May 7–10, 1998. Talk on: "The problem of classification of triangular maps with zero topological entropy."

[7] European Conference on Iteration Theory — ECIT 98, Muszyna, Poland, August 30–September 5, 1998. Invitation. Talk on: "The problem of classification of triangular maps with zero topological entropy."

[8] 28th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January 23–29, 2000. Talk on: "The problem of classification of triangular maps with zero topological entropy."

[9] 4th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 22–28, 2000. Talk on: "The problem of classification of triangular maps with zero topological entropy."

[10] 29th Winter School in Abstract Analysis, Lhota nad Rohanovem, February 3–10, 2001. Talk on: "Triangular maps non-decreasing on the fibres."

[11] 5th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 18–23, 2001. Talk on: "Triangular maps non-decreasing on the fibres."

[12] 6th Czech-Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 9–16, 2002. Talk on: "Open problems concerning triangular maps."

References

[ACS] L. Alseda, M. Chas, J. Smítal, *On the structure of the ω-limit sets for continuous maps of the interval*, Inter. Journal of Bifurcation and Chaos, Vol. 9, No. **9** (1999), 1719–1729.

[BaS] F. Balibrea, J. Smítal, *A characterization of the set $\Omega(f) \setminus \omega(f)$ for continuous maps of the interval with zero topological entropy*, Real Analysis Exchange **21(2)** (1995/6), 622–628.

[BGKM] F. Blanchard, E. Glasner, S. Kolyada, A. Maass, *A note on Li-Yorke chaos*, J. fur die reine und angewandte Mathematik **547** (2002), 51–68.

[BC] L. S. Block, W. A. Coppel, *One-dimensional Dynamics*, Lecture Notes in Math., vol. 1513, Springer-Verlag, Berlin, 1992.

[B] R. Bowen, *Entropy for group endomorphism and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.

[BS] A. M. Bruckner, J. Smítal, *A characterization of ω-limit sets of maps of the interval with zero topological entropy*, Ergodic Theory & Dynam. Syst. **13** (1993), 7–19.

[C] José S. Cánovas, *Personal communication.*

[FSS] V. V. Fedorenko, A. N. Sharkovsky, J. Smítal, *Characterizations of weakly chaotic maps of the interval*, Proc. Amer. Math. Soc. **110** (1990), 141–148.

[FPS1] G. L. Forti, L. Paganoni, J. Smítal, *Strange triangular maps of the square*, Bull. Austral. Math. Soc. **51** (1995), 395–415.

[FPS2] G. L. Forti, L. Paganoni, J. Smítal, *Dynamics of homeomorphisms on minimal sets generated by triangular mappings*, Bull. Austral. Math. Soc. **59** (1999), 1–20.

[Ko1] Z. Kočan, *The problem of classification of triangular maps with zero topological entropy*, Annales Mathematicae Silesianae **13** (1999), 181–192.

[Ko2] Z. Kočan, *Triangular maps non-decreasing on the fibres*, Bull. Austral. Math. Soc. (submitted).

[K] S. F. Kolyada, *On dynamics of triangular maps of the square*, Ergodic Theory & Dynam. Syst. **12** (1992), 749–768.

[KMS] S. F. Kolyada, M. Misiurewicz, L. Snoha, *Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval*, Fundam. Math. **160**, No. 2 (1999), 161–181.

[KS] S. F. Kolyada, A. N. Sharkovsky, *On topological dynamics of triangular maps of the plane*, in Proceedings of ECIT-89, Batschuns, Austria, World Scientific, Singapore (1991), 177–183.

[M] M. Misiurewicz, *Structure of mappings of an interval with zero entropy*, Publ. Math. IHES **53** (1981), 5–16.

[SKSF] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak, V. V. Fedorenko, *Dynamics of one-dimensional mappings*, Mathematics and its Applications (Dordrecht), Kluwer Academic Publishers (1997).

[S1] P. Šindelářová, *A counterexample to a statement concerning Lyapunov stability*, Acta Math. Univ. Comen. **70** (2001), 265–268.

[S2] P. Šindelářová, *A zero topological entropy map for which periodic points are not a G_δ set*, Ergod. Th. & Dynam. Syst. **22** (2002), 947–949.

[S3] P. Šindelářová, *A counterexample to a statement concerning recurrent points*, Proc. Amer. Math. Soc. (to appear).

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer 2000.