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Hamiltonian operators and related structures

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Contents

1. Introduction

This thesis is based on three independent papers [44], [45] and [46]. Their common subject are Hamiltonian structures and associated Hamiltonian and bi-Hamiltonian evolutionary partial differential equations. All three papers constitute an integral part of the thesis.

The first paper "The Darboux coordinates for a new family of Hamiltonian operators and linearization of associated evolution equations" was published in *Nonlinearity* in 2011 (see [46]) as well as the second paper entitled "A complete list of conservation laws for non-integrable compacton equations of $K(m,m)$ type," published in 2013 (see [44]). The third paper "Low-order Hamiltonian operators having momentum" was published in the *Journal of Mathematical Analysis and Applications* in 2013 (see [45]). All of these three papers contain new results in the theory of Hamiltonian evolutionary partial differential equations.

Informally, a Hamiltonian structure for a system of PDEs is just an operator giving rise to a Lie algebra structure (the Poisson bracket) on a suitable space of functionals. Its significance stems *inter alia* from the fact that for a Hamiltonian system the Poisson bracket of two conserved quantities is again a conserved quantity, see below for details. Moreover, as explained below, Hamiltonian structures provide a correspondence among conservation laws (and more broadly, cosymmetries) and symmetries. This correspondence plays an important role in the study of qualitative properties of the solutions of Hamiltonian systems (e.g. stability thereof) and frequently has deep consequences in physics.

The general theory of Hamiltonian structures (also known as Poisson structures) for finite-dimensional dynamical systems in their general form without relying on the canonical coordinates dates back to Sophus Lie (see e.g. the discussion in Chapter 6 of [29]). The extension to the case of systems of evolutionary PDEs goes back to the work of V.I. Arnol'd [1, 2] on the Hamiltonian structure of the Euler equations and to the discovery of Hamiltonian structure of the celebrated Korteweg–de Vries equation by C. Gardner [16]. Since then the subject underwent a very intense development which hardly lends itself to a reasonably concise description; see e.g. the monographs [4], [9], [10], [13], [19], [28], [29] and references therein. We only mention here the Dirac structures, an important and very broad generalization of Hamiltonian structures introduced by Dorfman [10].

The interest of international mathematical and mathematical physics communities in the theory of Hamiltonian and bi-Hamiltonian structures and the systems of partial differential equations related to these structures remains steadily high and was further increased by the recent advances in the field made by V. G. Kac et al. (see e.g. [7, 8] and references therein) using the Dirac structures and the so-called Poisson vertex algebras.

The bi-Hamiltonian systems, introduced by F. Magri [20], i.e., the systems admitting a pair of substantially different Hamiltonian operators (the so-called Hamiltonian pair), deserve special attention. Such systems are usually integrable, which in particular means that a sufficiently rich set of their solutions can be found. Bi-Hamiltonian systems occur in many areas, e.g. in mechanics, hydrodynamics, electrodynamics and others. This is one of the main reasons of interest in classification and study of the Hamiltonian operators: this can lead to new Hamiltonian pairs and hence to new integrable systems.

The theory of Hamiltonian and bi-Hamiltonian structures was developed by many mathematicians and physicists, in particular, V. I. Arnol'd, I. Dorfman, V. G. Drinfeld, B. A. Dubrovin, L. D. Faddeev, A. S. Fokas, I. M. Gel'fand, V. G. Kac, F. Magri, S. P. Novikov, P. J. Olver, V. E. Zakharov and others.

The two subsequent sections are of introductory nature and closely follow [29].

2. Basic definitions and notation

Throughout the thesis we work with Hamiltonian operators and Hamiltonian differential evolution equations in jet spaces. The introduction of jet spaces allows us to look at differential equations as algebraic ones. Below we shall not discuss the fairly sophisticated theory of jet bundles that occurs in the geometric theory of partial differential equations. Instead we define the jet space just in coordinates. Unless otherwise explicitly stated, all objects below will be assumed to be smooth.

Definition 1. Let $X \simeq \mathbb{R}^p$ and $U \simeq \mathbb{R}^q$ be vector spaces with the coordinates $x_1, \dots, x_p \in \mathbb{R}$ and $u^1, \dots, u^q \in \mathbb{R}$ respectively. Let $J = (j_1, \dots, j_k)$ with $1 \le j_k \le p$ be a multi-index of order $|J| = k$. Let $U_k \simeq \mathbb{R}^{q \cdot p_k}$, where $p_k = \begin{pmatrix} p+k-1 \\ k \end{pmatrix}$, be a vector space with the coordinates $u^\alpha_J, |J| = k, \alpha \in \{1, 2, \dots, q\}$. The space

$$X \times U^{(n)} = X \times U \times U_1 \times \cdots \times U_n$$

is then called *the jet space of the n-th order.*

Note that in the case $q = 1$, $p = 1$ the coordinates in the jet space of the n-th order are also denoted by $x, u, u_x, u_{xx}, u_{xxx}, \dots, u_{n \cdot x}$ or $x, u, u_1, u_2, u_3, \dots, u_n$.

The coordinates on the n-th order jet space represent the independent variables, the dependent variables and the derivatives of the dependent variables with respect to the independent variables up to the order n. The *n-th prolongation* of a function $f : X \to U$ is a function $\mathrm{pr}^{(n)} f : X \to U^{(n)}$, the coordinates of $\mathrm{pr}^{(n)} f(x)$ being $u^\alpha_J = \partial_J f^\alpha(x)$, where $\partial_J = \frac{\partial^k f^\alpha}{\partial x_{j_1} \dots \partial x_{j_k}}$ and $J = (j_1, \dots, j_k)$. In many cases it suffices to consider only an open subset $M^{(n)} = \{(x, u^{(n)}) \in X \times U^{(n)}, (x, u) \in M \subset X \times U\}$ of the n-th order jet space.

A smooth function $P = P(x_i, u^\alpha, u^\alpha_J)$ which depends on the independent variables, the dependent variables and finitely many derivatives of dependent variables with respect to independent variables is called a *differential function*. It is easily seen that each differential function is a function $P : M^{(k)} \to \mathbb{R}$, for some $k \in \mathbb{N} \cup \{0\}$, where the smallest such k is called the *order of the differential function* P and is denoted as $\mathrm{ord}\, P$. If we are not interested in how many derivatives P depends on we write simply $P = P[u]$. The set $\mathcal{A}$ of all differential functions carries the structure of a ring with the multiplication of differential functions.

Recall briefly several important operators defined on $\mathcal{A}$. *The total derivative* $D_{x_j} : \mathcal{A} \to \mathcal{A}$ *with respect to the independent variable* x_j is defined as

$$D_{x_j} := \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,j} \frac{\partial}{\partial u^{\alpha}_{J}},$$

where $u^{\alpha}_{J,j}$ is defined as $u^{\alpha}_{J,j} = \frac{\partial u^{\alpha}_{J}}{\partial x_j}$. Note that if $P = P[u]$ is a differential function then the sum $D_{x_j}(P)$ is actually finite, so no convergence issues arise. The *total derivative* $D_J : \mathcal{A} \to \mathcal{A}$ *with respect to a multi-index* $J = (j_1, \ldots, j_k)$ is defined as $D_J = D_{x_{j_1}} \circ D_{x_{j_2}} \circ \cdots \circ D_{x_{j_k}}$.

The Euler–Lagrange operator $E : \mathcal{A} \to \mathcal{A}^q$ is a q-tuple of operators $E = (E_1, \ldots, E_q)$, where each E_α is an operator $E_\alpha : \mathcal{A} \to \mathcal{A}$ defined as

$$E_\alpha := \sum_J (-D)_J \circ \frac{\partial}{\partial u^{\alpha}_{J}},$$

where $(-D)_J = (-1)^{|J|} \cdot D_{x_{j_1}} \circ D_{x_{j_2}} \circ \cdots \circ D_{x_{j_k}}$ for $J = (j_1, j_2, \ldots j_k)$. *The total divergence operator* is the operator $\mathrm{Div} : \mathcal{A}^p \to \mathcal{A}$ defined for any p-tuple of differential functions $(P_1, \ldots, P_p) \in \mathcal{A}^p$ as

$$\mathrm{Div}(P_1, \ldots, P_p) := D_{x_1}(P_1) + D_{x_2}(P_2) + \cdots + D_{x_p}(P_p).$$

Let $\mathfrak{D} = \sum_J P_J[u] D_J : \mathcal{A} \to \mathcal{A}$ be a linear operator. *The adjoint operator to the operator* $\mathfrak{D}$ is an operator $\mathfrak{D}^* : \mathcal{A} \to \mathcal{A}$ which satisfies (see e.g. [29])

$$\int_\Omega R \cdot \mathfrak{D}(S) \mathrm{d}x = \int_\Omega S \cdot \mathfrak{D}^*(R) \mathrm{d}x$$

for all differential functions $R, S \in \mathcal{A}$ that vanish for $u \equiv 0$, for every domain $\Omega \subset \mathbb{R}^p$ and every function $u = f(x)$ with compact support in Ω. By easy computations it can be verified that

$$\mathfrak{D}^* = \sum_J (-D)_J \circ P_J,$$

which means that for any differential function $Q \in \mathcal{A}$ the equality $\mathfrak{D}^*(Q) = \sum_J (-D)_J (P_J \cdot Q)$ holds.

The notion of the adjoint operator can be easily extended to the case of matrix differential operators in the following way. Let $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$ be a matrix differential operator with entries $\mathfrak{D}_{kl}$. Then the adjoint operator $\mathfrak{D}^* : \mathcal{A}^q \to \mathcal{A}^q$ is a matrix differential operator with entries $\mathfrak{D}^*_{kl} = (\mathfrak{D}_{lk})^*$.

Let $P = (P_1, \ldots, P_r) \in \mathcal{A}^r$ be an r-tuple of differential functions. The *Fréchet derivative* of the vector function P is a differential operator $\mathrm{D}_P : \mathcal{A}^q \to \mathcal{A}^r$ such that

$$\mathrm{D}_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left(P[u + \varepsilon Q[u]]\right).$$

It is easily seen that the operator D_P is a $q \times r$ matrix differential operator with entries

$$(\mathrm{D}_P)_{\mu\nu} = \sum_J \frac{\partial P_\mu}{\partial u^\nu_J} D_J,$$

where $\mu = 1, \dots, r$ and $\nu = 1, \dots, q$.

Now, any n-th order (smooth) system of differential equations can be viewed as a zero set of differential functions:

$$\widetilde{F}_\mu(x, u^{(n)}) = 0, \quad \mu = 1, \dots, l, \tag{1}$$

where $x = (x_1, \dots, x_p)$, $u = (u^1, \dots, u^q)$, and $u^{(n)}$ denotes the derivatives of u^αs with respect to x_js up to the order n. *A solution* of this system of differential equations is a smooth function $f(x)$ such that $\widetilde{F}_\mu(x, \mathrm{pr}^{(n)} f) = 0$ for all $\mu = 1, \dots, l$ whenever x lies in the domain of f.

A *generalized vector field* $\mathbf{v}$ is defined by the formula

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u] \frac{\partial}{\partial u^\alpha},$$

where $\xi^i[u], \phi_\alpha[u]$ are differential functions for all $i = 1, \dots, p$ and $\alpha = 1, \dots, q$. The *prolongation* of $\mathbf{v}$ is a formal sum

$$\mathrm{pr}\ \mathbf{v} = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J \phi^J_\alpha[u] \frac{\partial}{\partial u^\alpha_J},$$

where the sum is taken over all multi-indices J, and

$$\phi^J_\alpha = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i \right) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}.$$

Note that for any generalized vector field $\mathbf{v}$ its prolongation $\mathrm{pr}\ \mathbf{v} : \mathcal{A} \to \mathcal{A}$ is a derivation of the algebra $\mathcal{A}$ of differential functions. Again, no convergence issues arise because for any differential function $a \in \mathcal{A}$ the sum $\mathrm{pr}\ \mathbf{v}(a)$ is finite.

Definition 2 (see [29])**.** A generalized vector field $\mathbf{v}$ is *generalized symmetry of the system (1)* if

$$\mathrm{pr}\ \mathbf{v}(\widetilde{F}_\mu) = 0 \text{ for all } \mu = 1, \dots, l$$

on the solutions of (1).

Note that if all ξ^i and ϕ_α are functions of x and u only, $\mathbf{v}$ is the infinitesimal generator of the classical (Lie point) symmetry.

Let $\mathbf{v}$ be a generalized symmetry of (1). Define its *evolutionary representative*, i.e., a generalized vector field $\mathbf{v}_Q$ of the form $\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}$, where $Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i$. It is easy to prove (see [29]) that $\mathbf{v}$ is a generalized symmetry for (1) if and only if so is $\mathbf{v}_Q$. Moreover, both $\mathbf{v}$ and $\mathbf{v}_Q$ are equivalent in the sense that they differ by a trivial symmetry, i.e., by a generalized symmetry whose coefficients vanish on solutions of (1). Therefore when we are looking for symmetries of a given system we can without loss of generality restrict ourselves to the symmetries in evolutionary form. The main advantage of the evolutionary form of a generalized vector field is a simple form of its prolongation:

$$\mathrm{pr}\,(\mathbf{v}_Q) = \mathrm{pr}\left(\sum_{\alpha=1}^{q} Q_\alpha[u]\frac{\partial}{\partial u^\alpha}\right) = \sum_{\alpha,J} D_J(Q_\alpha)\frac{\partial}{\partial u_J^\alpha}.$$

The q-tuple of differential functions $Q = (Q_1,\ldots,Q_q)$ is called the *characteristic* of the generalized symmetry $\mathbf{v}_Q$. From now on by a *symmetry* we mean a generalized symmetry in the evolutionary form.

If we consider a system of evolution equations

$$u_t = F(x, \widetilde{u^{(n)}}), \tag{2}$$

where $u = (u^1,\ldots,u^q)$, $x = (x^1,\ldots,x^p)$, $F = (F_1,\ldots,F_q)$ is a q-tuple of differential functions, and $\widetilde{u^{(n)}}$ denotes the set of all derivatives of u up to order n not involving the differentiation w.r.t. the time t, then the symmetry condition (2) on $\mathbf{v}_Q$ can be rewritten in the form

$$D_t(Q_\nu) = \mathrm{pr}\ \mathbf{v}_Q(F_\nu) \text{ for all } \nu = 1\ldots,q,$$

where we now assume without loss of generality that $Q = Q(x,t,\widetilde{u^{(k)}})$ is free of the the derivatives of u involving the differentiation w.r.t. t.

The *Fréchet derivative* of a differential operator $\mathfrak{D} = \sum_K P_K[u]D_K$ with respect to an evolutionary vector field $\mathbf{v}_Q$ is the differential operator $\mathrm{pr}\ \mathbf{v}_Q(\mathfrak{D}) = \sum_K \mathrm{pr}\ \mathbf{v}_Q(P_K)D_K$.

A *conservation law* for the system (1) is the divergence expression

$$\mathrm{Div}(R) = 0, \tag{3}$$

which vanishes on all smooth solutions of (1), where $R = (R_1,\ldots,R_p)$ is a p-tuple of differential functions. For an evolutionary system (2), one of the independent variables, the variable labeled t which usually stands for the *time* (i.e., the evolution parameter), is naturally distinguished from the other independent variables. In this case the conservation law (3) can be written in the form

$$D_t(\rho) = \widetilde{\mathrm{Div}}(\sigma),$$

where the total divergence $\widetilde{\mathrm{Div}}$ on the right-hand side does not contain the total time derivative.

The differential function $\rho = \rho(x,t,\widetilde{u^{(k)}})$ is called *the density of a conservation law*, and the p-component differential vector function $\sigma = \sigma(x,t,\widetilde{u^{(s)}})$ is called the *flux*. It can be shown (see e.g. [29]) that for any $\Omega \subset X$ the functional $\mathcal{T}[t,u] = \int_\Omega \rho\ dx$ is a constant (depending on the solution!) for any given solution $u = f(x)$ of (2) such that $\sigma \to 0$ as $x \to \partial\Omega$.

It can be proved that the q-tuple $E(\rho)$, where E is the Euler-Lagrange operator, satisfies the equation

$$D_t(E(\rho)) + \mathrm{D}_F^*(E(\rho)) = 0,$$

which means that $E(\rho)$ is a *cosymmetry* for (2).

A *formal symmetry of order* m for the n-th order evolution system (2) in one independent variable x is a formal series

$$\mathfrak{S} = \sum_{j=-\infty}^{k} l_j D_x^j,$$

where the coefficients l_j are $q \times q$ matrices with differential functions (again free of the derivatives of u involving the differentiation w.r.t. t) as entries, such that

$$\deg\left(D_t(\mathfrak{S}) - [D_P, \mathfrak{S}]\right) \leq n + k - m.$$

Here the symbol deg stands for the degree of a formal series; recall that for $\mathfrak{M} = \sum_{j=-\infty}^{s} b_j D_x^j$ with $b_s \neq 0$ we have deg $\mathfrak{M} = s$ by definition, see e.g. [21, 22].

The procedure of finding formal symmetries whose coefficients do not explicitly depend on the time t for a given evolution system (2) is described in [21, 22, 23] both for the scalar case ($q = 1$) and the vector case ($q > 1$). Unfortunately, for the vector equations it is efficient only under certain technical assumptions on (2), see e.g. [21, 22] for details. On the other hand, the theory of time-independent formal symmetries for scalar evolution equations is essentially complete. The existence of formal symmetries is studied mainly using the so-called canonical densities.

Throughout the rest of this section we will consider only scalar equations ($q = 1$) in one independent variable x, i.e., equations of the form

$$u_t = F(x, u, u_1, \ldots, u_n), \tag{4}$$

where $F = F[u]$ is a differential function, and $u_j = \partial^j u / \partial x^j$.

Theorem 1 ([22, 23])**.** *Let $\mathfrak{S}$ be a time-independent formal symmetry of order $N > n$ and of the degree k for the equation (4). Then (4) possesses $N - n$ conserved densities*

$$\rho_i = \begin{cases} \operatorname{res}(\mathfrak{S}^{i/k}) & i \neq 0 \\ \operatorname{res}\, \log(\mathfrak{S}) & i = 0, \end{cases} \tag{5}$$

where $i = -1, 0, \ldots, N - n - 2$.

Recall that the *residue* and *logarithmic residue* of the formal series $\mathfrak{M} = \sum_{j=-\infty}^{k} l_j D_x^j$ are the coefficients res $\mathfrak{M} := l_{-1}$ and log res $\mathfrak{M} := l_{k-1}/l_k$, respectively.

It can be proved (see e.g. [21]) that a first few canonical densities (5) can be expressed also in terms of the coefficients of the operator D_F.

For instance, the first canonical densities for an n-th order evolution equation (4) with $n > 2$ are, up to a suitable choice of normalization, given by the formulas (see [21, 22, 23])

$$\rho_{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-1/n}, \tag{6}$$

$$\rho_0 = \frac{\partial F/\partial u_{n-1}}{\partial F/\partial u_n}. \tag{7}$$

This yields a criterion for existence of a formal symmetry of any fixed order and, as the following theorem implies, also provides a criterion for the existence of a time-independent generalized symmetry with the characteristic of the fixed order:

Theorem 2 (see [22, 23])**.** *Equation (4) possesses an explicitly time-independent formal symmetry of order $N > n$ if and only if the first $N-n$ canonical densities ρ_i, $i = -1, 0, 1, 2, \ldots, N-n-2$ are densities of local conservation laws.*

Existence of an explicitly time-independent formal symmetry of order $q > N$ is a necessary condition for (2) to possess explicitly time-independent generalized symmetries with the characteristic of order q.

Time-dependent generalized symmetries and time-dependent formal symmetries are treated *inter alia* in [41]. There the general form of a few leading terms for any time-dependent formal symmetry of order $r > n$ is given:

Theorem 3 (see [41])**.** *Any formal symmetry $\mathfrak{R}$ of (4) of degree k and of order $r > n$ can be written in the form*

$$\begin{gathered}\mathfrak{R} = \widetilde{\mathfrak{R}} + \sum_{j=k-n+1}^{k} d_j(t)\mathrm{D}_F^{j/n} + \frac{k}{n^2} d_k(t) D_x^{-1}(\Phi^{-1-1/n} D_t(\Phi))\mathrm{D}_F^{(k-n+1)/n} \\ + \frac{1}{n}\dot{d}_k(t) D_x^{-1}(\Phi^{-1/n})\mathrm{D}_F^{(k-n+1)/n},\end{gathered} \tag{8}$$

where $\Phi = \partial F/\partial u_n$, $d_i(t)$ are functions of t, and $\widetilde{\mathfrak{R}}$ is a formal series such that $\deg \widetilde{\mathfrak{R}} < k - n + 1$.

The rigorous definition of fractional powers of formal series used in (8) can be found in [22, 23].

In general, we see that there is an important relation among time-independent generalized symmetries and time-independent formal symmetries and canonical densities. This fact and the results from [41] are employed in section 6 below and in [44] in finding all time-independent generalized symmetries.

3. Hamiltonian operators and the associated Hamiltonian evolution equations

In this section we consider matrix differential operators in total derivatives $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$, their entries being differential operators of the form

$$\mathfrak{D}_{kl} = \sum_J P_{kl,J}[u] D_J,$$

where $P_{kl,J}[u]$ are arbitrary differential functions. We also give several equivalent conditions for the differential operators to be Hamiltonian.

Introduce the following equivalence relation on $\mathcal{A}$: Two differential functions P_1 and P_2 are *equivalent* if they differ by a total divergence of a differential vector function Q: $P_1 - P_2 = \mathrm{Div} Q$. The quotient space $\mathcal{A}/\mathrm{Div}$ is denoted by $\mathcal{F}$ and its members are called *functionals.* The equivalence class of a differential function P is denoted by $\int P \mathrm{d}x$.

The bracket associated with a differential operator $\mathfrak{D}$ is a $\mathbb{R}$-bilinear map $\{\cdot,\cdot\} : \mathcal{F}\times\mathcal{F} \to \mathcal{F}$ defined by the formula

$$\{\mathcal{P}, \mathcal{L}\} := \int \delta\mathcal{P} \cdot \mathfrak{D}(\delta\mathcal{L}) \mathrm{d}x, \tag{9}$$

where $\mathcal{P} = \int P \mathrm{d}x$ and $\mathcal{L} = \int L \mathrm{d}x$. The operator δ in (9) is the *variational derivative* $\delta : \mathcal{F} \to \mathcal{A}^q, \delta : \mathcal{P} = \int P \mathrm{d}x \mapsto E(P)$.

Definition 3. (see [29]) A linear differential operator in total derivatives $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$ is said to be *Hamiltonian* if the associated bracket $\{\cdot,\cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is Poisson, i.e., it satisfies the following conditions:

(1) $\{\mathcal{R}, \mathcal{L}\} = -\{\mathcal{L}, \mathcal{R}\}$ (*skew symmetry*)
(2) $\{\{\mathcal{P}, \mathcal{L}\}, \mathcal{R}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{L}\} + \{\{\mathcal{L}, \mathcal{R}\}, \mathcal{P}\} = 0$ (*the Jacobi identity*).

The definition of Hamiltonian operators by means of Poisson brackets is conceptually transparent but it is rather difficult to use in concrete computations. Conditions equivalent to the skew-symmetry condition and the Jacobi identity condition of Poisson brackets were studied by Olver [29], Dorfman [10], and others. Several of them are listed below.

Theorem 4 (see [29])**.** *Let $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$ be a matrix differential operator. Then the associated bracket $\{\cdot,\cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is skew-symmetric if and only if the operator $\mathfrak{D}$ is skew-adjoint, i.e., $\mathfrak{D}^* = -\mathfrak{D}$.*

Theorem 5 (see [29])**.** *Let $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$ be a skew-adjoint matrix differential operator. Then the associated bracket $\{\cdot,\cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ satisfies the Jacobi identity condition if and only if the condition*

$$\int [P \cdot (\mathrm{pr}\ \mathbf{v}_{\mathfrak{D}(Q)}(\mathfrak{D}))(R) + Q \cdot (\mathrm{pr}\ \mathbf{v}_{\mathfrak{D}(R)}(\mathfrak{D}))(P) + R \cdot (\mathrm{pr}\ \mathbf{v}_{\mathfrak{D}(P)}(\mathfrak{D}))(Q)] \mathrm{d}x = 0$$

holds for all q-tuples of differential functions $P, Q, R \in \mathcal{A}^q$.

For example, any skew-adjoint matrix differential operator whose entries have constant coefficients or quasi-constant coefficients is always a Hamiltonian one. This readily follows from the above theorem.

The condition in the following theorem was obtained by Dorfman in [10] only for translation-invariant Hamiltonian operators in one independent variable. It can be shown that it remains valid for a broader class of operators:

Theorem 6 (cf. [10])**.** *The skew-adjoint matrix differential operator* $\mathfrak{D} : \mathcal{A}^q \to \mathcal{A}^q$ *is Hamiltonian if and only if the following condition holds for arbitrary* $Q, R \in \mathcal{A}^q$*:*

$$(D_{\mathfrak{D}}Q)\mathfrak{D}R - (D_{\mathfrak{D}}R)\mathfrak{D}Q = \mathfrak{D}(D_{\mathfrak{D}}R)^{*}Q, \tag{10}$$

where for any $Q \in \mathcal{A}^q$ *the operator* $D_{\mathfrak{D}}Q : \mathcal{A}^q \to \mathcal{A}^q$ *is defined by the formula*

$$(D_{\mathfrak{D}}Q)R := \mathrm{pr}\ \mathbf{v}_R(\mathfrak{D})(Q).$$

We now define Hamiltonian evolution equations related to a given Hamiltonian operator $\mathfrak{D}$. For a given functional $\mathcal{H} = \int H\mathrm{d}x$ they are precisely the equations of the Hamiltonian flow associated to the Hamiltonian evolution vector field with the characteristic $\mathfrak{D}(\delta\mathcal{H})$:

Definition 4. Let $u_t = Q[u]$ be an evolution system. We say that this system is Hamiltonian with respect to a matrix Hamiltonian differential operator $\mathfrak{D}$ with a Hamiltonian functional $\mathcal{H}$ if the right-hand side Q of our system can be written as

$$Q[u] = \mathfrak{D}(\delta\mathcal{H}).$$

Example 1. (see e.g. [29]) The Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x$$

can be written in Hamiltonian form in two ways, the first being

$$u_t = D_x\left(u_{xx} + \frac{1}{2}u^2\right) = \mathfrak{D}_1(\delta\mathcal{H}_1),$$

where $\mathfrak{D}_1 = D_x$ is a constant-coefficient operator, and hence a Hamiltonian one, and $\mathcal{H}_1 = \int(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3)\mathrm{d}x$, and the second being

$$u_t = \left(D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x\right)u = \mathfrak{D}_0(\delta\mathcal{H}_0),$$

where $\mathfrak{D}_0 = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x$ is a Hamiltonian operator (now this is not that aparent) and $\mathcal{H}_0 = \int \frac{1}{2}u^2\mathrm{d}x$.

Recall that a functional $\mathcal{P} = \int P\mathrm{d}x$ is a *conserved quantity* (or an *integral of motion*) for an evolution system (2) if P is a conservation law density for (2).

The Hamiltonian systems enjoy the following important additional property:

Theorem 7 (see e.g. [29])**.** *If* $\mathcal{P} = \int P\mathrm{d}x$ *and* $\mathcal{P} = \int Q\mathrm{d}x$ *are conserved quantities for a Hamiltonian system* $u_t = \mathfrak{D}(\delta\mathcal{H})$ *then their Poisson bracket* $\{\mathcal{P}, \mathcal{Q}\}$ *is again a conserved quantity for the system under study.*

This means that, at least in principle, we can obtain new conserved quantities by taking the Poisson brackets of the known ones.

Definition 5. We say that linearly independent Hamiltonian differential operators $\mathfrak{D}_1$ and $\mathfrak{D}_2$ form a *Hamiltonian pair* (or that they are *compatible*) if every $\mathbb{R}$-linear combination $a\mathfrak{D}_1 + b\mathfrak{D}_2$ is also a Hamiltonian operator. An evolution equation which can be written in Hamiltonian form in two ways so that the Hamiltonian operators in question are compatible is said to be *bi-Hamiltonian.*

It is important to stress that under certain fairly minor technical assumptions bi-Hamiltonian systems are completely integrable in the sense that they have infinitely many conserved quantities that Poisson commute with respect to *both* Poisson brackets associated with $\mathfrak{D}_1$ and $\mathfrak{D}_2$, see e.g. [4], [10], [8] and [29] for further details.

Example 2. It can be proved that the Hamiltonian operators $\mathfrak{D}_1$ and $\mathfrak{D}_0$ from Example 1 form a Hamiltonian pair. Hence, the Korteweg–de Vries equation is a bi-Hamiltonian equation.

We now take a somewhat closer look at the symmetries and conservation laws of Hamiltonian evolution systems. Let $\mathfrak{D}$ be a Hamiltonian operator and let us consider the associated Hamiltonian system

$$u_t = \mathfrak{D}(\delta\mathcal{P}), \tag{11}$$

where $\mathcal{P}$ is a functional. Recall that a functional $\mathcal{C}$ is called a *Casimir functional*, if $\mathfrak{D}(\delta\mathcal{C}) = 0$. In terms of the Poisson bracket, a functional $\mathcal{C}$ is a Casimir functional if and only if $\{\mathcal{C}, \mathcal{H}\} = 0$ for all $\mathcal{H} \in \mathcal{F}$. Recall that a *Hamiltonian vector field* associated to a given Hamiltonian operator $\mathfrak{D}$ and a functional $\mathcal{H}$ is the unique vector field $\hat{\mathbf{v}}_{\mathcal{H}}$ which satisfies

$$\hat{\mathbf{v}}_{\mathcal{H}}(\mathcal{P}) = \{\mathcal{P}, \mathcal{H}\} \quad \text{for all functionals} \quad \mathcal{P} \in \mathcal{F}.$$

It can be verified that such a Hamiltonian vector field has the characteristic equal to $\mathfrak{D}\delta\mathcal{H}$.

Note that a Hamiltonian operator $\mathfrak{D}$ yields a Lie algebra homomorphism $\mathfrak{D} \circ \delta$ from the Lie algebra of functionals endowed with the Poisson bracket associated with $\mathfrak{D}$ to the Lie algebra of evolutionary vector fields, see e.g. [4] or [29] for details.

There is a nice Noether-type correspondence between symmetries and conservation laws for Hamiltonian systems:

Theorem 8 (see e.g. [29])**.** *Let* $u_t = \mathfrak{D}\delta\mathcal{H}$ *be a Hamiltonian system of evolution equations. A Hamiltonian vector field* $\hat{\mathbf{v}}_{\mathcal{P}}$ *with the characteristic* $\mathfrak{D}\delta\mathcal{P}$ *determines a generalized symmetry of the system under study if and only if there is an equivalent functional* $\tilde{\mathcal{P}} = \mathcal{P} - \mathcal{C}$, *differing from $\mathcal{P}$ by a time-dependent Casimir functional $\mathcal{C}[t,u]$ such that $\tilde{\mathcal{P}}$ is a conserved quantity and thus defines a conservation law.*

4. The classification results concerning Hamiltonian operators in one independent variable x and one dependent variable u

In this section we give a brief survey of the results concerning the classification of Hamiltonian operators in one independent variable x and one dependent variable u obtained respectively by Olver [31], Cooke [6] and, more recently, by de Sole, Kac and Wakimoto [7]. Because of the nonlinear Jacobi identity condition the classification of Hamiltonian operators is quite difficult. It should be mentioned that to date the classification of Hamiltonian operators of this type is available only up to the order 11 (recall that since a Hamiltonian operator must be skew-adjoint its order is always an odd number), where in the case of the operators of the 1st, 3rd and 5th order the general formulas for them are known whereas the operators of the 7th, 9th and 11th order are described only modulo contact transformations. We also provide a discussion of the so-called Darboux coordinates for Hamiltonian operators in one dependent and one independent variable. In the last part of this section we survey some known results on Hamiltonian operators that possess momentum.

Theorem 9 ([25]). *Let $\mathfrak{D}_1$ be a Hamiltonian operator in the variables x, u. Under the differential substitution*

$$x = \varphi(y, v, v_1, \ldots, v_m), \quad u = \psi(y, v, v_1, \ldots, v_n), \tag{12}$$

where $v_j = D_y^j(v)$, and D_y is the total derivative with respect to y, the operator $\mathfrak{D}_1$ goes into the Hamiltonian operator $\mathfrak{D}_2$ defined by the formula

$$\overline{\mathfrak{D}}_1 = (D_y(\varphi))^{-1}\mathfrak{K}^* \circ \mathfrak{D}_2 \circ \mathfrak{K}, \tag{13}$$

where

$$\mathfrak{K} = \sum_{i=0}^{\max(m,n)} (-1)^i D_y^i \circ \left(\frac{\partial\psi}{\partial v_i} D_y(\varphi) - \frac{\partial\varphi}{\partial v_i} D_y(\psi)\right),$$

$\mathfrak{K}^*$ is the formal adjoint of $\mathfrak{K}$, and $\overline{\mathfrak{D}}_1$ is obtained from $\mathfrak{D}_1$ upon using (12) and setting $D_x = (D_y(\varphi))^{-1} D_y$.

Note that in general a differential substitution may be non-invertible. However, there is a subclass of the class of differential substitutions which has a group structure, namely, the pseudogroup of contact transformations.

Definition 6. *A contact transformation* is a transformation of the form

$$x = \varphi(y, v, v_y),\ u = \psi(y, v, v_y), D_x = \frac{1}{D_y\varphi} D_y,$$

which satisfies the following conditions:

$$\frac{\partial\varphi}{\partial v_y}D_y\psi = \frac{\partial\psi}{\partial v_y}D_y\varphi, \text{ and } D_y\varphi \text{ and } \rho = \frac{\partial\psi}{\partial v} - \frac{\partial\varphi}{\partial v}\frac{D_y\psi}{D_y\varphi} \text{ are nonzero differential functions.}$$

A *special contact transformation* is a contact transformation of the form (see e.g. [25])

$$x = \varphi(y, v, v_y) = y + w(v, v_y) \text{ and } u = \psi(v, v_y).$$

As it was said above, the set of all contact transformations is a pseudogroup with respect to the composition. The set of all special contact transformations also forms a pseudogroup.

Theorem 10 (see [25])). *A differential substitution preserves the order of a scalar local Hamiltonian operator if and only if it is contact.*

Note that in general the operator $\mathfrak{D}_2$ may contain nonlocal terms unless (12) is a contact transformation, cf. e.g. [3, 7, 25].

Before we treat general forms of Hamiltonian operators recall the notion of the so-called *level* $\operatorname{lev}\mathfrak{D}$ of a Hamiltonian operator $\mathfrak{D} = \sum_{i=0}^k P_i[u]D_x^i$ which is defined as $\max_i\{i+\operatorname{ord} P_i\}$. It is proved (see [7]) that, with the exception of $m = 1$ in the case $N = 3$, the only possible values m of the level of a non-quasiconstant-coefficient Hamiltonian operator of order $N \le 11$ are $m = N, N+1, N+2$ (we say that $\mathfrak{D} = \sum_{i=0}^k P_i[u]D_x^i$ is a *quasiconstant-coefficient operator* if its coefficients depend only on x).

The general form of the first-order Hamiltonian operators was originally found by Dorfman and Gel'fand in [17] but there it was done only for Hamiltonian operators that did not explicitly depend on the space variable x. Their result was extended by Olver in [31]. The general forms of the third- and fifth- order Hamiltonian operators were found by D. B. Cooke in [6]. We present here only those results that are relevant for us, i.e., mainly the results concerning the leading coefficients and the general form of a fifth-order Hamiltonian operator with the leading coefficient equal to 1, which (in the case of fifth-order Hamiltonian operators) will be important for us in section 7 below and also in [45]. The conditions on the coefficients of a fifth-order operator in order for the latter to be Hamiltonian are listed in that section for a certain special case.

Theorem 11. *The following assertions hold:*

(1) (see [31]) Any first-order Hamiltonian operator $\mathfrak{D}$ must be of the form

$$\mathfrak{D} = \frac{1}{E(a)} \circ D_x \circ \frac{1}{E(a)},$$

where $a = a(x, u, u_x)$.

(2) (see [6]) Any third-order Hamiltonian operator $\mathfrak{D}$ must be of the form

$$\mathfrak{D} = \frac{1}{f} \circ \left(D_x \circ \frac{1}{f}\right)^3 + \text{ lower-order terms,}$$

where $f = \alpha u_{xx} + \beta$, $\alpha = \alpha(x, u, u_x)$ and $\beta = \beta(x, u, u_x)$.

(3) (see [6]) Any fifth-order Hamiltonian operator $\mathfrak{D}$ must be of the form

$$H = \frac{1}{f} \circ \left[D_x \circ \frac{1}{f} \right]^5 + \text{ lower-order terms,}$$

where $f = \alpha u_{xx} + \beta$ and $\alpha = \alpha(x, u, u_x)$ and $\beta = \beta(x, u, u_x)$.

Theorem 12 (see [6]). *A fifth-order Hamiltonian operator whose leading coefficient is* 1 *must be of the form*

$$\mathfrak{D} = D_x^5 + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where b and c are functions of x alone. Otherwise they are given by

$$\begin{aligned} b &= \frac{3}{2}(u+\alpha)^{-1}(u_{xx} + \alpha'') - \frac{7}{4}(u+\alpha)^{-2}(u_x + \alpha')^2 + \beta(u+\alpha) + \gamma, \\ c &= -\frac{z_4}{z} + \frac{\beta z_1^2}{2z} + \frac{w z_2}{2z} - \frac{w z_1^2}{4z^2} - \frac{w_1 z_1}{z} + \frac{9 z_1 z_3}{2z^2} - \frac{129 z_1^2 z_2}{8z^3} + \frac{273 z_1^4}{32 z^4} \\ &\quad + \frac{33 z_2^2}{8z^2} - \frac{\beta z_2}{2} - \frac{3z\beta''}{2} - \frac{\beta' z_1}{2} - \frac{\beta^2 z^2}{2} + \frac{w^2}{2}, \end{aligned}$$

where α, β, and γ are functions of x only, w and z are given by

$$w = \beta z + \gamma, \;\; z = u + \alpha,$$

and $w_i = D_x^i(w)$, $z_i = D_x^i(z)$.

If $\beta = 0$, then any choice of α and γ yields a Hamiltonian operator.

If $\beta \neq 0$, then

$$\gamma = -\frac{\rho}{\beta^2} - \frac{\beta''}{2\beta} + \frac{(\beta')^2}{4\beta^2},$$

where ρ is an arbitrary constant.

As we have already mentioned at the beginning of this section, the classification of scalar Hamiltonian operators of the 7th, 9th and 11th order was obtained using the Poisson vertex algebras by de Sole, Kac, and Wakimoto in [7] modulo contact transformations, i.e., the authors defined the following equivalence on the set of all scalar Hamiltonian operators of a given order: two Hamiltonian operators $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are *equivalent*, if there exists a contact transformation which turns $\mathfrak{D}_1$ into $\mathfrak{D}_2$. Namely, de Sole, Kac and Wakimoto [7] have found a complete list of canonical forms for 7th-, 9th- and 11th-order Hamiltonian operators modulo contact transformations and gave a conjectural list of such normal forms for all odd orders.

Moreover, in [7], a new family of compatible Hamiltonian operators

$$H^{(N,0)} = D_x^2 \circ \left(\frac{1}{u} D_x \right)^{2n} \circ D_x, \;\; N = 2n+3, n = 0, 1, 2, \ldots$$

was introduced and the following conjecture was made:

Conjecture 1 (see [7])**.** *For any translation-invariant Hamiltonian operator H of order $N \geq 7$ there exists a contact transformation that turns H into either a quasiconstant coefficient skew-adjoint differential operator or into a linear combination of the operators $H^{(j,0)}$ with $3 \leq j \leq N$, j odd.*

This conjecture could be a starting point in the solution of one of the most important problems in the theory of infinite-dimensional Hamiltonian systems, the problem of finding the Darboux coordinates; for more details see section 5 below and [46]. It is basically the question of whether, given a Hamiltonian operator $\mathfrak{D}$, there exist canonical coordinates such that $\mathfrak{D}$ takes some simple form, usually the form of the *Gardner operator* D_x. This is analogous to the case of finite-dimensional Hamiltonian systems where such coordinates do exist by the celebrated Darboux theorem. The existence of these coordinates (we still call them the *Darboux coordinates* to stress the analogy with the finite-dimensional case) for the first-order Hamiltonian operators in one independent and one dependent variable was established by Olver; the same results for the third- and fifth- order Hamiltonian operators in one independent and one dependent variable were obtained by Cooke [6]. For higher-order Hamiltonian operators the problem is still open.

A quite important property of Hamiltonian operators in one independent and one dependent variable is the existence of momentum.

Definition 7. Let $\mathfrak{D}$ be a Hamiltonian operator in one independent variable x and one dependent variable u. We say that $\mathfrak{D}$ *has momentum* if there exists a functional $\mathcal{P}$ such that $\mathfrak{D}(\delta\mathcal{P}) = u_x$.

Note that this definition readily extends to the case of several dependent variables.

The existence of momentum for a given operator $\mathfrak{D}$ can be employed [25] e.g. for averaging the corresponding Hamiltonian system $u_t = \mathfrak{D}(\delta\mathcal{H})$ or for finding the traveling wave solutions of the form $u(x-at)$ for the Hamiltonian system $u_t = \mathfrak{D}(\delta\mathcal{H})$ in the following way: let $\mathfrak{D}$ be a Hamiltonian operator that possesses momentum, i.e., there exists a functional $\mathcal{P}$ such that $\mathfrak{D}(\delta\mathcal{P}) = u_x$. Let $u = u(x-at)$ be a solution of the Hamiltonian equation $u_t = \mathfrak{D}(\delta\mathcal{H})$. The function $u(x-at)$ satisfies

$$u_t + au_x = 0.$$

Therefore, it also satisfies the equation

$$\mathfrak{D}(\delta\mathcal{H} + \delta a\mathcal{P}) = \mathfrak{D}\delta(\mathcal{H} + a\mathcal{P}) = 0. \tag{14}$$

The above equation (14) is equivalent to a lower-order one:

$$\delta(\mathcal{H} + a\mathcal{P}) = \delta\mathcal{C}, \tag{15}$$

where $\mathfrak{C}$ is the (most general) Casimir functional for $\mathfrak{D}$. It can be shown that for $\mathfrak{D} = \sum_{i=0}^{k} p_i[u]D_x^i$ we have $\delta\mathfrak{C} = Q(x, u)$. With this in mind equation (15) is usually much easier to solve than (14).

Operators having momentum could be also employed for the construction of hierarchies of local symmetries (or higher commuting flows) in the following fashion.

Suppose we are given a Hamiltonian operator in one dependent variable u and one independent variable x, say $\mathfrak{D}$, possessing momentum, which means that there exists a functional $\mathcal{P} = \int h\mathrm{d}x$ such that $u_x = \mathfrak{D}(\delta\mathcal{P})$. Further assume that there exists another translation-invariant Hamiltonian operator $\mathfrak{E}$ which is compatible with $\mathfrak{D}$ and that the operator $\mathfrak{R} := \mathfrak{E} \circ \mathfrak{D}^{-1}$ is a weakly nonlocal hereditary operator. Then $\mathfrak{R}$ is a recursion operator for the equation $u_{t_0} = u_x$ and, under a further minor technical assumption of normality of $\mathfrak{R}$ in the sense of [39], by Theorem 1 from [39] the quantities $\mathfrak{R}^i(u_x)$ are local and the associated flows commute for all $i = 1, 2, 3, \ldots$, and we thus have an infinite hierarchy of local commuting flows $u_{t_j} = \mathfrak{R}^j(u_x)$, $j = 0, 1, 2, \ldots$.

The classification of Hamiltonian operators having momentum was initiated by Mokhov [25]. The first result in this direction is the following proposition:

Proposition 1 (see [25]). *Let $\mathfrak{D}$ be a Hamiltonian operator that possesses momentum. Then $\mathfrak{D}$ is translation-invariant.*

Mokhov [25] has also established that the existence of momentum is preserved by the special contact transformations, so one can perform the classification modulo the latter. He succeeded in classifying all first- and third-order Hamiltonian operators having momentum:

Theorem 13. (see [25])

(1) A first-order Hamiltonian operator has momentum if and only if it is translation-invariant.

(2) An arbitrary translation-invariant Hamiltonian operator of the third order can be reduced by a special contact transformation to one of the operators (16)-(18).

  (a) An operator

$$\mathfrak{D} = \pm\frac{1}{u_x}\left[D_x^3 + 2SD_x + D_xS\right] \circ \frac{1}{u_x} + 2fD_x + D_xf, \tag{16}$$

  where $S = \frac{u_3}{u_1} - \frac{3}{2}\frac{(u_2)^2}{(u_1)^2}$ and f is an arbitrary function of u only, has momentum. The corresponding functional is of the form $\int p(u)\mathrm{d}x$, where $p(u)$ is the solution of the equation

$$\pm\frac{\partial^4 p}{\partial u^4} + 2f(u)\frac{\partial^2 p}{\partial u^2} + \frac{\partial p}{\partial u}\frac{\partial f}{\partial u} - 1 = 0.$$

  (b) An operator

$$\mathfrak{D} = \pm\left[D_x^3 + 2AuD_x + Au_x\right], \; A = \mathrm{const} > 0 \tag{17}$$

has momentum.

(c) An operator

$$\mathfrak{D} = \pm\left[D_x^3 + AD_x\right], A = \text{const.} \tag{18}$$

does not have momentum.

Our results on the classification of fifth-order Hamiltonian operators having momentum are treated in section 7 below and further details can be found in [45].

5. The Darboux coordinates for a new family of Hamiltonian operators and linearization of associated evolution equations.

In the paper [46] we consider a recently introduced family of compatible Hamiltonian operators from [7] of the form $H^{(N,0)} = D^2 \circ ((1/u) \circ D)^{2n} \circ D$, where $N = 2n+3$, $n = 0, 1, 2, \ldots$. For each of these operators we found a differential substitution which turns this operator into a simpler form. In Theorem 14 we present a transformation which simultaneously turns the operators $H^{(N,0)}$ for all $N \geq 3$ into operators with constant coefficients:

Theorem 14 (see [46]). *The transformation $x = v$, $u = 1/v_y$ turns the N-*th *order Hamiltonian operator $H^{(N,0)} = D_x^2 \circ ((1/u) \circ D_x)^{2n} \circ D_x$, where $N = 2n+3$, $n = 0, 1, 2, \ldots,$ into the Hamiltonian operator* with constant coefficients $\widetilde{H}^{(N,0)} = -D_y^{2n+1}$.

Note that the transformation $x = v, u = 1/v_y$ can be realized as a composition of the potentiation $x = z$, $u = w_z$ and the hodograph transformation $z = v$, $w = y$, the latter transformation being contact (and actually even a point one).

We can further strengthen the result of Theorem 14 and find for each operator $H^{(N,0)}$ with $N \geq 3$ the corresponding Darboux coordinates:

Corollary 1 (see [46]). *The transformation $x = (-1)^{\frac{n+1}{2}} w_n$, $u = (-1)^{\frac{n+1}{2}}/w_{n+1}$ where $w_k = D_z^k(w)$, and z is the new independent variable maps the Hamiltonian operator $H^{(N,0)} = D_x^2 \circ ((1/u) \circ D_x)^{2n} \circ D_x$ into the first-order Gardner operator D_z for any odd $N \geqslant 3$.*

Bringing a Hamiltonian operator into the Gardner form enables us to render the associated Hamiltonian systems into the canonical Hamiltonian form and construct Lagrangian representations (modulo potentialization) for these systems [32].

These results can be employed for linearization of potential forms of bi-Hamiltonian equations

$$u_t = \mathfrak{D}_1 \delta_u \mathfrak{T}_1 = \mathfrak{D}_2 \delta_u \mathfrak{T}_2, \tag{19}$$

where $\mathfrak{D}_i = \sum_{j=1}^{k_i} c_{ij} H^{(N_{ij},0)}$, $i = 1, 2$, k_i are arbitrary natural numbers and c_{ij} are arbitrary constants:

Proposition 2 (see [46])**.** *The transformation $z = v$, $w = y$, where y is the new independent variable, linearizes the potential form of the bi-Hamiltonian evolution equation (19). The potential form of (19) reads*

$$w_t = \check{\mathfrak{D}}_1 \delta_w \check{\mathfrak{T}}_1 = \check{\mathfrak{D}}_2 \delta_w \check{\mathfrak{T}}_2, \tag{20}$$

where $\check{\mathfrak{D}}_i = \sum_{j=1}^{k_i} c_{ij} \check{H}^{(N_{ij},0)}$, $i = 1, 2$, $\check{H}^{(N,0)} = -D_z \circ \left(\frac{1}{w_z} D_z\right)^{2n}$, *and* $\check{\mathfrak{T}}_i$ *are obtained from* $\mathfrak{T}_i$ *using the substitution* $x = z$, $u = w_z$.

Example 3. Consider a bi-Hamiltonian evolution equation

$$u_t = D_x^3 \left(u^{-2}\right) = H^{(3,0)} \delta_u \mathfrak{T}_1 = H^{(5,0)} \delta_u \mathfrak{T}_2, \tag{21}$$

where $\mathfrak{T}_1 = -\int \mathrm{d}x/u$ and $\mathfrak{T}_2 = \int x^2 u \mathrm{d}x$.

The potential form (20) of (21) reads

$$w_t = D_z^2 \left(w_z^{-2}\right) = \check{H}^{(3,0)} \delta_w \check{\mathfrak{T}}_1 = \check{H}^{(5,0)} \delta_w \check{\mathfrak{T}}_2. \tag{22}$$

Recall that $u = w_z$ and $x = z$; we have $\check{H}^{(3,0)} = -D_z$, $\check{H}^{(5,0)} = -D_z \circ ((1/w_z)D_z)^2$, $\check{\mathfrak{T}}_1 = -\int \mathrm{d}z/w_z$, and $\check{\mathfrak{T}}_2 = \int z^2 w_z \mathrm{d}z$. Note that (22) has, up to a rescaling of t, the form (2.31) from [5].

In perfect agreement with Proposition 2 (cf. also Proposition 2.2 in [5]) the hodograph transformation $z = v$, $w = y$ linearizes (22) into a (trivially) bi-Hamiltonian equation

$$v_t = -2v_{yyy} = \widetilde{H}^{(3,0)} \delta_v \widetilde{\mathfrak{T}}_1 = \widetilde{H}^{(5,0)} \delta_v \widetilde{\mathfrak{T}}_2, \tag{23}$$

where $\widetilde{H}^{(3,0)} = -D_y$, $\widetilde{H}^{(5,0)} = -D_y^3$, $\widetilde{\mathfrak{T}}_1 = -\int v_y^2 \mathrm{d}y$, and $\widetilde{\mathfrak{T}}_2 = \int v^2 \mathrm{d}y$.

The following corollary now easily follows from the previous results:

Corollary 2 (see [46])**.** *The differential substitution $x = v$, $u = 1/v_y$ relates any equation of the form (19) to a linear evolution equation with constant coefficients.*

The results concerning the transformation properties of the operators $H^{(N,0)}$ can be used as a first step in the proof of the existence of Darboux coordinates for any Hamiltonian operator. The proof uses, among other results, the following lemma:

Lemma 1. *Given a quasiconstant skew-adjoint differential operator $\mathfrak{D} = \sum_{i=0}^{N} a_i(x) D_x^i$ there exists a transformation of the form $x = y, u = \sum_{i=0}^{k} b_i(x) v_i$ that turns $\mathfrak{D}$ into the Gardner operator D_y.*

Now, using the statement of Theorem 14 and the fact that the transformation from this theorem is a composition of the potentiation and a contact transformation, Conjecture 1 can be restated as follows:

Conjecture 2 (see [46]). *Any translation-invariant Hamiltonian operator of order $N \geq 7$ can be transformed into the Gardner operator D by either a contact transformation or a composition thereof with the transformation $x = v, u = 1/v_y$ and a transformation of the form $x = y, u = \sum_{i=0}^{k} a_i(x) v_i$.*

6. A complete list of conservation laws for non-integrable compacton equations of $K(m,m)$ type

In the paper [44] we consider a family of the so-called generalized $K(m, n)$ equations,

$$u_t = aD_x^3(u^n) + bD_x(u^m), \quad a, b, \in \mathbb{R}, \tag{24}$$

that are of great interest both for mathematicians and physicists since they describe many natural wave phenomena with finite span. This family of differential equations in a slightly less general form first appeared in [37] in 1993 as a family of differential equations whose solutions are compactons, i.e., solitons with compact support. Below and in [44] we restrict ourselves only to the case $m = n$.

If $m = n$ then the $K(m, n)$ equations are easily seen to be Hamiltonian with respect to the Hamiltonian operator $\mathfrak{D} = aD_x^3 + bD_x$, the Hamiltonian functional being $\mathcal{H} = \int \int u^m du dx$. Thus, equation (24) for $m = n$ can be written as

$$u_t = aD_x^3(u^m) + bD_x(u^m) = \mathfrak{D}\delta\mathcal{H}. \tag{25}$$

Our goal in [44] is to find all local conservation laws for all non-integrable cases of $K(m, m)$ equations, and, in particular, to give a rigorous proof of the result of Olver (see [37]) who found four conservation laws for the $K(2, 2)$ equation and claimed without proof that no other conservation law for this equation exist. This is done using the so-called symmetry approach to integrability, see the discussion in section 2 and e.g. [21, 22, 23] for details. The relevant results employed by us below were given in Theorems 2 and 3.

Computing the canonical density ρ_{-1} from (6) for a generalized $K(m, m)$ equation we see that it is a conserved density only for the cases $m = -2, -1/2, 0, 1$. For all $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ the first canonical density is not a conserved density. The cases $m = -2$ and $m = -1/2$ were identified by Rosenau as integrable cases of $K(m, m)$ equations and for $m = 0$ and $m = 1$ the $K(m, m)$ equation is just linear. Therefore we can stop computing canonical densities at this stage and notice that the $K(m, m)$ equations for $m \in \mathbb{R} \setminus \{-2, -1/2, 0, 1\}$ have no generalized time-independent symmetries of order greater than 3, and hence, in particular, these equations are not symmetry integrable.

Proposition 3 (see [44]). *If $m \neq -2, -1/2, 0, 1$, then the corresponding generalized $K(m, m)$ equation (25) has no* explicitly time-independent *generalized symmetries of order greater than 3; in particular, equation (25) is not symmetry integrable.*

Proposition 3 ensures non-existence of *explicitly time-independent* generalized symmetries of order greater than 3. However, this result can be further strengthened using Theorem 1 from [41] to cover also explicitly time-dependent symmetries:

Proposition 4 (see [44]). *If $m \neq -2, -1/2, 0, 1$, then the corresponding generalized $K(m,m)$ equation (25) has no generalized symmetries of order greater than 3, including explicitly time-dependent ones.*

Proof. For the equations under study we obviously have $D_t(\rho_{-1}) \notin \mathrm{Im} D_x$ and $\rho_{-1} \notin \mathrm{Im} D_x$, where $\rho_{-1} = (anu^{n-1})^{-1/3}$. This fact in conjunction with Theorem 3 implies that for any (even possibly explicitly time-dependent) formal symmetry $\mathfrak{R}$ of order $r > 3$ we have $d_k(t) = 0$ (see (8)), as the coefficients of a formal symmetry must be differential functions, and hence the coefficients at the nonlocal terms $D_x^{-1}(\Phi^{-4/3}D_t(\Phi))$ and $D_x^{-1}(\Phi^{-1/3})$ must vanish (in our case $\Phi = anu^{n-1}$, see the definition in Theorem 3). But the equality $d_k(t) = 0$ means that the leading term of our formal symmetry $\mathfrak{R}$ must vanish, i.e., we arrive at a contradiction. This means that the equations under study have no formal symmetries, time-dependent or not, of order greater than three, and therefore (cf. e.g. the discussion at the end of section 2 in [41]) they also have no generalized symmetries (be they time-dependent or not) with the charcteristics of order greater than three. □

Now finding all generalized symmetries becomes just a straightforward computation:

Proposition 5 (see [44]). *The only generalized symmetries of (25) for $m \neq -2, -1/2, 0, 1$ are those with the characteristics $Q_1 = u_x$, $Q_2 = u_t$ and $Q_3 = (m-1)tu_t + u$, i.e., x- and t-translations and the scaling symmetry.*

Now we can find all local conservation laws for all non-integrable cases of generalized $K(m,m)$ equations. If ρ is a density of a conservation law of an evolution equation, then the function $\gamma = E(\rho)$ is a cosymmetry of the equation in question. Now we use the Hamiltonian structure of any $K(m,m)$ equation and the fact that Hamiltonian operator $\mathfrak{D}$ turns cosymmetries of the equation $u_t = \mathfrak{D}\delta\mathcal{H}$ into symmetries, whence we infer that $\mathfrak{D}(E(\rho))$ is a symmetry and, since we know that $\mathrm{ord}(\mathfrak{D}(E(\rho))) \leq 3$, we have $\mathrm{ord}\, E(\rho) = 0$. Thus, we arrive at the following result:

Theorem 15 (see [44]). *If ρ is a density of a local conservation law for a generalized $K(m,m)$ equation, where $m \neq -2, -1/2, 0, 1$, then it is, up to the addition of a trivial density, a function of x, t and u only.*

After straightforward computations we obtain

Theorem 16 (see [44]). *The only local conservation laws of the form $D_t(\rho) = D_x(\sigma)$ for the generalized $K(m,m)$ equation (25) with $m \neq -2, -1/2, 0, 1$, are, modulo the addition of* *trivial conservation laws, just the linear combinations of the four conservation laws which for $b \neq 0$ are given by the formulas*

$$\rho_1 = \int u^m du \qquad \sigma_1 = \left(mau_{xx}u^{2m-1} + \frac{am(m-2)}{2}u_x^2 u^{2m-2} + \frac{b}{2}u^{2m} \right)$$

$$\rho_2 = u \qquad \sigma_2 = aD_x^2(u^m) + bu^m$$

$$\rho_3 = u \sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) \quad \sigma_3 = aD_x^2(u^m)\sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) - \sqrt{ab}D_x(u^m)\cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right)$$

$$\rho_4 = u \cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) \quad \sigma_4 = aD_x^2(u^m)\cos\left(\frac{\sqrt{b}}{\sqrt{a}}x\right) + \sqrt{ab}D_x(u^m)\sin\left(\frac{\sqrt{b}}{\sqrt{a}}x\right).$$

If $b = 0$, then the conservation law with the density ρ_3 is trivial, and the densities ρ_2 and ρ_4 coalesce. However, there are two other conservation laws in such a case, namely

$$\rho_5 = xu \quad \sigma_5 = aD_x^2(xu^m) - 3aD_x(u^m)$$
$$\rho_6 = x^2u \quad \sigma_6 = aD_x^2(x^2u^m) + 6au^m - aD_x(xu^m),$$

i.e., for $b = 0$ equation (25) with $m \neq -2, -1/2, 0, 1$ also has, up to the addition of trivial conservation laws, just four conservation laws with the densities $\rho_1, \rho_2, \rho_5, \rho_6$ and the fluxes $\sigma_1, \sigma_2, \sigma_5, \sigma_6$.

If a and b have different signs then sines and cosines of a complex variable appear in the formulas for ρ_3, ρ_4, σ_3 and σ_4. In this case it is convenient to divide ρ_3 by the imaginary unit i and use the following *real* densities and fluxes instead of the above ρ_3, ρ_4, σ_3 and σ_4:

$$\tilde{\rho}_3 = cu \sinh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right) \quad \tilde{\sigma}_3 = caD_x^2(u^m)\sinh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right) - \sqrt{|ab|}D_x(u^m)\cosh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right)$$

$$\tilde{\rho}_4 = u \cosh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right) \quad \tilde{\sigma}_4 = aD_x^2(u^m)\cosh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right) - c\sqrt{|ab|}D_x(u^m)\sinh\left(\frac{\sqrt{|b|}}{\sqrt{|a|}}x\right),$$

where $c = 1$ if $a > 0$ and $b < 0$, and $c = -1$ if $a < 0$ and $b > 0$.

The conserved functional corresponding to the first conserved density is the energy, i.e., the integral of motion associated with the invariance under the time shifts. If $m = 2k - 1$ where $k \in \mathbb{Z} \setminus \{0, 1\}$, then the fact that the quantity $\int u^{m+1}dx$ is conserved immediately implies the following property of the solutions of the corresponding $K(m, m)$ equation: if a solution $u(x, t)$ of (25) belongs to the space $L^{2k}(\mathbb{R})$, i.e., $\int_{\mathbb{R}} |u|^{2k}dx < \infty$, at the time $t = t_0$ then $u(x, t) \in L^{2k}(\mathbb{R})$ for all $t \geq t_0$.

7. Low-order Hamiltonian operators having momentum

The paper [45] addresses the problem of classification of Hamiltonian operators possessing momentum. Namely, there we employ special contact transformations to classify fifth-order Hamiltonian differential operators in one dependent and one independent variable possessing momentum. As it was mentioned in section 3, this problem for first- and third- order Hamiltonian operators was solved by O. Mokhov in [25]. Up to a special contact transformation he classified all first- and third- order translation invariant Hamiltonian operators having momentum. His results are listed in Theorem 13.

Let us stress that throughout the rest of this section we tacitly restrict ourselves to Hamiltonian differential operators in one dependent and one independent variable.

In [45] we build on the work of Mokhov and, using special contact transformations which preserve the existence of momentum, describe all fifth-order Hamiltonian operators that possess momentum. Like Mokhov, we look for such operators among the translation-invariant ones. At the first step we classify all fifth-order translation-invariant Hamiltonian operators according to their leading coefficient up to special contact transformations:

Proposition 6 (see [45])**.** *Any fifth-order translation-invariant Hamiltonian operator can be reduced by a special contact transformation to an operator with the leading coefficient equal to either* ± 1 *or* $\pm\frac{1}{u_1^4}$.

At the next step we find general forms of fifth-order translation-invariant Hamiltonian operators with the leading coefficients 1, $1/u_1^4$ and $-1/u_1^4$. The general form of any fifth-order Hamiltonian operator with the leading coefficient equal to 1 was obtained earlier by D. Cooke in [6], see Proposition 12 above.

We should have also been given here a general form of a translation-invariant fifth-order Hamiltonian operator with the leading coefficient equal to -1 as well. However, there is a special contact transformation $x = y, u = \mathrm{i}v$ which turns any operator with the leading coefficient -1 into an operator with the leading coefficient 1, and we will prove below that no fifth-order Hamiltonian operator with the leading coefficient equal to 1 possesses momentum. Therefore, as special contact transformations preserve the property of existence (or non-existence) of momentum, it is readily seen that also no fifth-order Hamiltonian operator with the leading coefficient equal to -1 has momentum. Hence, the case of the leading coefficient equal to -1 is not interesting for us, and we can leave it aside.

The conditions from [6] on the coefficients of a general fifth-order Hamiltonian operator $\mathfrak{D} = aD_x^5 + D_x^5 \circ a + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c$ with the leading coefficient equal to $\frac{1}{u_1^4}$, i.e., the case $a = \frac{1}{2u_1^4}$ are the following

$$\begin{aligned}
\frac{\partial c}{\partial u_6} &= 0 \\
\frac{\partial b}{\partial u_4} &= 0 \\
\frac{\partial c}{\partial u_5} &= -\frac{3}{u_x^5} \\
\frac{\partial b}{\partial u_3} &= \frac{5}{u_1^5}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial c}{\partial u_4} &= \frac{1}{3u_1^6}\left(85u_2 - 2\frac{\partial b}{\partial u_2}u_1^6\right) \\
\frac{\partial c}{\partial u_3} &= \frac{1}{3u_1^7}\left(-16\frac{\partial b}{\partial u_2}u_1^6u_2 - 225u_1u_3 - 9D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^7 + 410u_2^2 + 6bu_1^6\right) \\
\frac{\partial b}{\partial u_1} &= \frac{1}{3u_1^7}\left(26\frac{\partial b}{\partial u_2}u_1^6u_2 + 340u_1u_3 + 7D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^7 - 550u_2^2 - 6bu_1^6\right) \\
\frac{\partial c}{\partial u_2} &= \frac{1}{6u_1^8}\left(-3\frac{\partial b}{\partial u}u_1^8 + 140\frac{\partial b}{\partial u_2}u_1^6u_2^2 + 80bu_1^6u_2 + 11390u_1u_2u_3 - 96\frac{\partial b}{\partial u_2}u_1^7u_3 - 14260u_2^3\right. \\
&\left. -27D_x^2\left(\frac{\partial b}{\partial u_2}\right)u_1^8 + 21D_x(b)u_1^7 - 1200u_1^2u_4 + 2\frac{\partial b}{\partial u_2}bu_1^{12} - 82D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^7u_2\right) \\
\frac{\partial c}{\partial u_1} &= \frac{1}{6u_1^9}\left(18\frac{\partial b}{\partial u}u_1^8u_2 - 42D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^7u_2^2 + 416\frac{\partial b}{\partial u_2}u_1^6u_2^3 - 271D_x(b)u_1^7u_2 - 856bu_1^6u_2^2\right. \\
&+80bu_1^7u_3 + 14730u_1^2u_2u_4 - 214D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^8u_3 - 68\frac{\partial b}{\partial u_2}u_1^8u_4 - 136D_x^2\left(\frac{\partial b}{\partial u_2}\right)u_1^8u_2 \\
&+2D_x\left(\frac{\partial b}{\partial u_2}\right)bu_1^{13} - 4\frac{\partial b}{\partial u_2}D_x(b)u_1^{13} - 92450u_1u_2^2u_3 - 1080u_1^3u_5 - 21D_x^3\left(\frac{\partial b}{\partial u_2}\right)u_1^9 \\
&\left.+3D_x^2(b)u_1^8 + 7610u_1^2u_3^2 - 3D_x\left(\frac{\partial b}{\partial u}\right)u_1^9 - 404\frac{\partial b}{\partial u_2}u_1^7u_2u_3 - 4\frac{\partial b}{\partial u_2}bu_1^{12}u_2 + 87920u_2^4\right) \\
\frac{\partial c}{\partial u} &= \frac{1}{6u_1^{10}}\left(-21D_x^4\left(\frac{\partial b}{\partial u_2}\right)u_1^{10} + 9D_x^2\left(\frac{\partial b}{\partial u}\right)u_1^{10} + 28410u_1^3u_3u_4 + 15730u_1^3u_2u_5 - 66D_x^2(b)u_1^8u_2\right. \\
&-608D_x(b)u_1^7u_2^2 - 717D_x(b)u_1^8u_3 - 13280bu_1^6u_2^3 - 660bu_1^8u_4 - 205510u_1^2u_2u_3^2 - 139340u_1^2u_2^2u_4 \\
&+838900u_1u_2^3u_3 - 80D_x(b)\frac{\partial b}{\partial u_2}u_1^{13}u_2 - 1416\frac{\partial b}{\partial u_2}u_1^7u_2^2u_3 - 2062D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^8u_2u_3 \\
&-464\frac{\partial b}{\partial u_2}bu_1^{12}u_2^2 - 32\frac{\partial b}{\partial u_2}bu_1^{13}u_3 - 120D_x\left(\frac{\partial b}{\partial u_2}\right)bu_1^{13}u_2 - 744\frac{\partial b}{\partial u_2}u_1^8u_2u_4 - 673120u_2^5 \\
&+440cu_1^8u_2 - 6D_x(b)bu_1^{13} - 232b^2u_1^{12}u_2 - 1092u_1^4u_6 - 9D_x^3(b)u_1^9 + 6D_x(c)u_1^9 \\
&-252D_x^3\left(\frac{\partial b}{\partial u_2}\right)u_1^9u_2 - 956D_x^2\left(\frac{\partial b}{\partial u_2}\right)u_1^8u_2^2 - 408D_x^2\left(\frac{\partial b}{\partial u_2}\right)u_1^9u_3 + 8c\frac{\partial b}{\partial u_2}u_1^{14} + 6\frac{\partial b}{\partial u}bu_1^{14} \\
&-4b^2\frac{\partial b}{\partial u_2}u_1^{18} - 1304D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^7u_2^3 - 282D_x\left(\frac{\partial b}{\partial u_2}\right)u_1^9u_4 - 2200\frac{\partial b}{\partial u_2}u_1^6u_2^4 - 68\frac{\partial b}{\partial u_2}u_1^9u_5 \\
&+66D_x\left(\frac{\partial b}{\partial u}\right)u_1^9u_2 + 192\frac{\partial b}{\partial u}u_1^8u_2^2 + 12\frac{\partial b}{\partial u}u_1^9u_3 - 12D_x^2\left(\frac{\partial b}{\partial u_2}\right)bu_1^{14} - 12D_x\left(\frac{\partial b}{\partial u_2}\right)D_x(b)u_1^{14} \\
&\left.-708\frac{\partial b}{\partial u_2}u_1^8u_3^2 + 3124bu_1^7u_2u_3\right).
\end{aligned}
$$

Solving this complicated system of differential equations and a very similar system for the case $a = -\frac{1}{2u_1^4}$ we arrive at the following lemma.

Lemma 2 (see [45]). *A fifth-order translation-invariant Hamiltonian operator whose leading coefficient is* $\pm 1/u_1^4$ *must be of the form*

$$\mathfrak{D} = \pm\frac{1}{2u_1^4}D_x^5 \pm D_x^5 \circ \frac{1}{2u_1^4} + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6}\left(\pm 10u_3u_1 \mp 55u_2^2 + 2\alpha u_1^4\right), \\ c &= \frac{1}{u_1^8}\left(3u_1^6u_2\frac{\partial\alpha}{\partial u} + 2u_1^5u_3\alpha - 6u_1^4u_2^2\alpha + \beta u_1^8 \mp 3u_1^3u_5 \pm 65u_1^2u_2u_4 \pm 50u_1^2u_3^2\right. \\ &\quad \left.\mp 615u_1u_2^2u_3 \pm 735u_2^4\right), \end{aligned}$$

and α *and* β *are functions of* u *only.*

Remark 1. It can be shown that there is no special contact transformation which preserves the leading coefficient and simultaneously eliminates one of the unknown functions α, β.

Now turn to the property of having momentum. Notice that the Fréchet derivative of the variational derivative of an arbitrary functional is a self-adjoint differential operator, see e.g. [29]. The following proposition states that any differential function h whose Fréchet derivative is a self-adjoint operator and which satisfies the condition $\mathfrak{D}(h) = u_1$, where $\mathfrak{D}$ is a fifth-order Hamiltonian operator with the leading coefficient of differential order less than or equal to 1, is of the form $h = h(x, u)$. It can be easily verified that any differential function of this form is the variational derivative of the functional $\mathcal{P} = \int\int h(x, u)\mathrm{d}u\mathrm{d}x$. Thus, instead of looking for a functional $\mathcal{P}$ such that $\mathfrak{D}\delta_u\mathcal{P} = u_1$ it suffices to check the existence of a differential function $h(x, u)$ that satisfies the condition $\mathfrak{D}(h) = u_1$.

Proposition 7 (see [45]). *Let* $\mathfrak{D}$ *be a fifth-order Hamiltonian operator whose leading coefficient is of differential order less than or equal to 1,*

$$\mathfrak{D} = aD_x^5 + D_x^5 \circ a + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c, \ \mathrm{ord}(a) \leq 1.$$

If there is a differential function $h[u]$ *such that* $\mathfrak{D}(h) = u_x$ *and* $\mathrm{D}_h = (\mathrm{D}_h)^*$, *then* $h = h(x, u)$.

As the last step we use the above proposition and obtain the following results:

Proposition 8 (see [45]). *No fifth-order Hamiltonian operator with the leading coefficient* ± 1 *has momentum.*

Proposition 9 (see [45]). *Any fifth-order translation-invariant Hamiltonian operator with the leading coefficient* $\pm 1/u_1^4$ *has momentum, and the corresponding functional* $\mathcal{P}$ *is of the form* $\mathcal{P} = \int\int h(u)\mathrm{d}u\mathrm{d}x$, *where* $h(u)$ *is a solution of the equation*

$$\pm\frac{\partial^5 h}{\partial u^5} + 2\alpha(u)\frac{\partial^3 h}{\partial u^3} + 3\frac{\partial\alpha(u)}{\partial u}\frac{\partial^2 h}{\partial u^2} + 3\frac{\partial^2\alpha(u)}{\partial u^2}\frac{\partial h}{\partial u} + 2\beta(u)\frac{\partial h}{\partial u} + \frac{\partial^3\alpha(u)}{\partial u^3}h + \frac{\partial\beta(u)}{\partial u}h - 1 = 0, \quad (26)$$

where $\alpha(u)$ *and* $\beta(u)$ *are as in Lemma 2.*

Combining Propositions 8 and 9 with the fact that special contact transformations preserve existence of momentum, we arrive at our main result.

Theorem 17 (see [45])**.** (1) *A fifth-order Hamiltonian operator which is not translation-invariant cannot have momentum.*

(2) *A fifth-order translation-invariant Hamiltonian operator that can be transformed using a special contact transformation into an operator with the leading coefficient* ± 1 *cannot have momentum.*

(3) *Any fifth-order translation-invariant Hamiltonian operator that can be transformed using a special contact transformation into an operator with the leading coefficient* $\pm 1/u_1^4$ *has momentum.*

Example 4 (see [45])**.** The operator

$$\mathfrak{D} = \frac{1}{2u_1^4} D_x^5 + D_x^5 \circ \frac{1}{2u_1^4} + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6}\left(10u_3u_1 - 55u_2^2 + u_1^4\right), \\ c &= \frac{1}{u_1^8}\left(u_1^5u_3 - 3u_1^4u_2^2 - u_1^8 - 3u_1^3u_5 + 65u_1^2u_2u_4 + 50u_1^2u_3^2 - 615u_1u_2^2u_3 + 735u_2^4\right), \end{aligned}$$

is of the form from Lemma 2 ($\alpha = 1/2$, $\beta = -1$). The function $h(u) = -u/2$ is a solution of the ordinary differential equation

$$\frac{\partial^5 h}{\partial u^5} + \frac{\partial^3 h}{\partial u^3} - 2\frac{\partial h}{\partial u} - 1 = 0.$$

The functional $\mathcal{P} = -\frac{1}{4}\int u^2 \mathrm{d}x$ satisfies the condition $\mathfrak{D}\delta_u \mathcal{P} = u_1$.

Example 5 (see [45])**.** The operator

$$\mathfrak{D} = \frac{1}{2u_1^4} D_x^5 + D_x^5 \circ \frac{1}{2u_1^4} + bD_x^3 + D_x^3 \circ b + cD_x + D_x \circ c,$$

where

$$\begin{aligned} b &= \frac{1}{2u_1^6}\left(10u_3u_1 - 55u_2^2 + 2\sin(u)u_1^4\right), \\ c &= \frac{1}{u_1^8}\left(3u_1^6u_2\cos(u) + 2u_1^5u_3\sin(u) - 6u_1^4u_2^2\sin(u) + (\sin(u)+u)u_1^8 - 3u_1^3u_5\right. \\ &\quad \left. + 65u_1^2u_2u_4 + 50u_1^2u_3^2 - 615u_1u_2^2u_3 + 735u_2^4\right), \end{aligned}$$

is of the form from Lemma 2 ($\alpha = \sin(u)$, $\beta = \sin(u) + u$). The function $h(u) = 1$ is a solution of the ordinary differential equation

$$\frac{\partial^5 h}{\partial u^5} + 2\sin(u)\frac{\partial^3 h}{\partial u^3} + 3\cos(u)\frac{\partial^2 h}{\partial u^2} - \sin(u)\frac{\partial h}{\partial u} + 2\frac{\partial h}{\partial u} + h - 1 = 0,$$

and hence the functional $\mathcal{P} = \int u \, \mathrm{d}x$ satisfies the condition $\mathfrak{D}\delta_u \mathcal{P} = u_1$.

To conclude, we give an outline of the algorithm enabling one to decide whether a given translation-invariant fifth-order Hamiltonian operator has momentum.

Algorithm. Let a fifth-order translation-invariant Hamiltonian operator $\mathfrak{D} : \mathcal{A} \to \mathcal{A}$ be given.

Step 1 If its leading coefficient depends on u_{xx}, then express the leading coefficient in the form

$$\frac{\pm 1}{(\alpha(u,u_x)u_{xx} + \beta(u,u_x))^6}.$$

Use the special contact transformation

$$x = y + w(v, v_y),\ u = \psi(v, v_y)$$

which is the inverse to the special contact transformation

$$y = x + \tilde{w}(u, u_x), v = \tilde{\psi}(u, u_x),$$

where $\tilde{w}$ nonzero and $\frac{\partial \tilde{w}}{\partial u_x} \not\equiv 0$, and $\tilde{w}(u, u_x)$ and $\tilde{\psi}(u, u_x)$ are functions that satisfy the following two differential equations:

$$\frac{\partial \tilde{\psi}}{\partial u} u_x + 1 = \frac{\alpha}{\beta} \frac{\partial \tilde{\psi}}{\partial u_x},\ (1 + D_x(\tilde{w})) \frac{\partial \tilde{\psi}}{\partial u_x} = \frac{\partial \tilde{w}}{u_x} D_x(\tilde{\psi}).$$

In this way we reduce the fifth-order Hamiltonian operator to the operator whose leading coefficient depends at most on u_x.

Step 2 Now we consider a Hamiltonian operator with the leading coefficient that depends at most on u_x. According to [6], this leading coefficient can be written in the form

$$\frac{\pm 1}{(\alpha(u)u_x + \beta(u))^4}.$$

Step 3 If $\beta \not\equiv 0$ then the operator $\mathfrak{D}$ does not possess momentum. If $\beta \equiv 0$ then the Hamiltonian operator possesses momentum, and the associated functional $\mathcal{P}$ can be found using the approach described earlier in this section.

8. Presentations related to the thesis

(1) IXth International Workshop Lie Theory and its Applications in Physics, Varna, Bulgaria, June 20 - 26, 2011.
Talk: The Darboux coordinates for a new family of Hamiltonian operators and linearization of associated evolution equations

(2) Tenth Workshop on Interactions between Dynamical Systems and Partial Differential Equations 2012, Barcelona, Spain, May 28 - June 1, 2012.
Talk: Low-order Hamiltonian operators having momentum.

(3) XXth International Conference on Integrable Systems and Quantum Symmetries, Prague, Czech Republic, June 17 - 23, 2012
Talk: A complete list of conservation laws for non-integrable compacton equations of $K(m, m)$ type.
(4) 8th International Conference Algebra Geometry Mathematical Physics, Brno, Czech Republic, September 12 - 14, 2012.
Talk: A complete list of conservation laws for non-integrable compacton equations of $K(m, m)$ type.
(5) 15th International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, June 7 - 12, 2013.
Talk: A complete list of conservation laws for non-integrable compacton equations of $K(m, m)$ type.
(6) International Conference on Moduli, Operads, Dynamics, Kongsberg, Norway, July 9 - 12, 2013.
Talk: Low-order Hamiltonian operators having momentum.

9. Publications constituting the body of the thesis

[1] J. Vodová, *The Darboux coordinates for new family of Hamiltonian operators and linearization of associated evolution equations*, Nonlinearity 24 (2011), 2569–2574.
[2] J. Vodová, *A complete list of conservation laws for non-integrable compacton equations of K(m, m) type*, Nonlinearity 26 (2013), no. 3, 757–762.
[3] J. Vodová, *Low-order Hamiltonian operators having momentum*, J. Math. Anal. Appl. 401 (2013), no. 2, 724–732.

10. Papers citing the publications constituting the body of the thesis

[1] D. Talati, R. Turhan, *On a Recently Introduced Fifth-Order Bi-Hamiltonian Equation and Trivially Related Hamiltonian Operators*, SIGMA 7 (2011) paper 081.

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