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Three types of chaos on discrete dynamical systems

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1 Introduction

The Thesis is based on four independent papers [1] – [4]. Their common subject are discrete dynamical systems generated by continuous maps of a compact metric space.

In the first part we give an answer for a natural question: what can be said about a "size" (in a sense of measure) of ω-scrambled set?

In the second part we study relations between Li and Yorke chaos and ω-chaos and their dependence on the cardinality of the system's domain.

The third part provides two counterexamples which disprove that two points Li and Yorke chaos implies uncountable Li and Yorke chaos on a general compact metric space. The first one is constructed on Cantor sets and the second one on an arcwise connected compactum.

In the fourth part we refute the main result from [WChL] and we extend the main result from the first part.

Finally, in the fifth part we summarize all implications between distributional chaos, Li and Yorke chaos and ω-chaos on an interval, a circle and a general compact metric space.

2 Basic terminology and notation

Let (X, d) be a compact metric space and $C(X)$ the set of all continuous maps $f : X \to X$. By $f^n(x)$ we denote the n-th iteration of x under f. The sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$ and $f^{n+1}(x) = f^n(f(x))$, is called the *trajectory* of x under f. The set $\omega_f(x)$ of all accumulation points of the trajectory is the *ω-limit set* of x under f.

A map $f \in C(X)$ is *(topologically) transitive* if for any nonempty open sets $U, V \subset X$ there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$; f is *bitransitive* if f^2 is transitive.

In the Thesis we consider the following three types of chaos: distributional chaos, Li and Yorke chaos and ω-chaos introduced by [SS], [LY] and [Li], respectively.

For an f in the class $C(X)$, for $x, y \in X$, $t \in \mathbb{R}$ and a positive integer n, let

$$\xi(x, y, n, t) = \#\{i;\ 0 \le i < n \text{ and } d(f^i(x), f^i(y)) < t\},$$

where $\#A$ denotes the cardinality of the set A. Put

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, n, t) \quad \text{and}$$

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, n, t).$$

Then both F^*_{xy} and F_{xy} are nondecreasing maps, with $0 \le F_{xy} \le F^*_{xy} \le 1$, $F^*_{xy}(t) = 0$ for $t < 0$, and $F_{xy}(t) = 1$ for $t > \operatorname{diam}(X)$. We refer to F^*_{xy} and F_{xy} as the *upper* and *lower distribution map* of x and y, respectively. The map f is *distributionally chaotic* (briefly, *dC*) if there is a set $D \subset X$ containing at least two points such that for any $x \ne y$ in D, $F_{xy} < F^*_{xy}$ (by this we mean that $F_{xy}(t) < F^*_{xy}(t)$ for all t in an interval), this set is called a *d-scrambled set* for the map f.

A set $S \subset X$ containing at least two points is called an *LY-scrambled set for f* if for any two $x \ne y$ in S is

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \quad \text{and}$$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

The map f is *Li and Yorke chaotic* (briefly, *LYC*) if there is an uncountable LY-scrambled set. Stronger notions of Li and Yorke chaos are these with *two points*, *infinite* or with an *uncountable* LY-scrambled set. To distinguish between these three types of Li and Yorke chaos we use the notation LY_2C, $LY_\infty C$ or LY_uC, respectively. Moreover, f is *extremely LYC* if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) = \operatorname{diam}(X).$$

A set $\Omega \subset X$ containing at least two points is called an ω^u-*scrambled set for* f if, for any two $x \neq y$ in S,

1. $\omega_f(x) \setminus \omega_f(y)$ is uncountable,

2. $\omega_f(x) \cap \omega_f(y) \neq \emptyset$ and

3. $\omega_f(x) \setminus Per(f) \neq \emptyset$.

The map f is ω^u-*chaotic* (briefly, $\omega^u C$) if there is an uncountable ω^u-scrambled set. In particular, f is $\omega_2^u C$, $\omega_\infty^u C$ or $\omega_u^u C$ if ω^u-scrambled set contains two, infinitely many, or uncountably many points, respectively.

The first condition from the definition of the ω^u-scrambled set could not be fulfilled if the space X is infinite. Therefore we introduce the following definition.

A set $\Omega \subset X$ containing at least two points is called an ω^∞-*scrambled set for* f if, for any two $x \neq y$ in S,

1. $\omega_f(x) \setminus \omega_f(y)$ is infinite,

2. $\omega_f(x) \cap \omega_f(y) \neq \emptyset$ and

3. $\omega_f(x) \setminus Per(f) \neq \emptyset$.

The map f is ω^∞-*chaotic* (briefly, $\omega^\infty C$) if there is an infinite ω^∞-scrambled set. In particular, f is $\omega_2^\infty C$ or $\omega_\infty^\infty C$ if ω^∞-scrambled set contains two or infinitely many points, respectively.

Let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of a compact metric space X, and $f \in C(X)$. Let $M(X, f)$ be the set of invariant Borel measures of f. Recall that a $\mu \in M(X, f)$ is *ergodic* (also f is called ergodic) if the only members B of $\mathcal{B}(X)$ with $f^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$. The map f is called *uniquely ergodic* if and only if $M(X, f)$ is a singleton. For more details see [Wa].

A subset $M \subset X$ is *minimal* if and only if for each $x \in M$, $\omega_f(x) = M$. A point x in X is *recurrent* under f if $x \in \omega_f(x)$, the set of all recurrent points under f is denoted by $U(f)$. Moreover, if $\omega_f(x)$ is minimal then the point x in X is *uniformly recurrent* under f and the set of all uniformly recurrent points under f is denoted by $UR(f)$.

Denote by Σ_2 the set of all sequences $x = x_1x_2x_3\dots$ where $x_n \in \{0, 1\}$ for each n, equipped with the metric of pointwise convergence. By a *shift* map we mean the map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x_1x_2x_3\dots) = x_2x_3x_4\dots$. This map is continuous (see, e.g., [Fu]).

A subset C of a compact metric space X is called a *Cantor set* if and only if it is nonempty, bounded, totally disconnected and perfect (e.g., the Cantor ternary set). By a *continuum* we mean a compact, connected set which contains more than one point. Finally, denote by I the unit closed interval.

3 Scrambled sets for transitive maps

The measure of LY-scrambled sets of $f \in C(I)$ was studied by many authors. In [S1] there is given an example of a function whose LY-scrambled set has full outer Lebesgue measure, maps in [Ka] and [S2] have LY-scrambled sets with positive Lebesgue measure, [BH] and [Mi] give examples of function LY-chaotic almost everywhere. Babilonová in [Ba1] (resp. [BS]) improved these results by showing that any bitransitive continuous map of the interval is conjugate to a map extremely LY-chaotic (resp. distributionally chaotic) almost everywhere.

The following Theorem A gives an answer for a natural question: What can be said about a "size" (in a sense of measure) of its ω-scrambled set?

Theorem A. *Every bitransitive $f \in C(I)$ is conjugate to $g \in C(I)$, which satisfies the following conditions:*

(i) there is a c-dense ω-scrambled set for g (hence g is ω-

chaotic),

(ii) *there is an extremely LY-scrambled set for g with full Lebesgue measure (consequently, g is extremely LY-chaotic),*

(iii) *every ω-scrambled set of g has zero Lebesgue measure.*

4 Relations between LY-chaos and ω-chaos

It is obvious, that $\mathrm{LY}_u\mathrm{C} \Rightarrow \mathrm{LY}_\infty\mathrm{C} \Rightarrow \mathrm{LY}_2\mathrm{C}$; the converse implications are true for continuous maps on the interval [KS] or on the circle [Ku] but, on general compact metric spaces they are no more valid [HY], [FPS1], [3]. Also $\omega^u_u\mathrm{C} \Rightarrow \omega^u_\infty\mathrm{C} \Rightarrow \omega^u_2\mathrm{C} \Rightarrow \omega^\infty_2\mathrm{C}$ and $\omega^u_\infty\mathrm{C} \Rightarrow \omega^\infty_\infty\mathrm{C} \Rightarrow \omega^\infty_2\mathrm{C}$, and again it is possible to show (see examples in [2]) that the converse implications are not true in the general case. There are four examples, constructed in [2], which describe relations between these two notions ($\mathrm{LY}_\bullet\mathrm{C}$ and $\omega^\bullet_\bullet\mathrm{C}$), see the table bellow. These are examples of continuous maps on several spaces with different types of LYC and ωC.

cardinality of the space X	type of $\mathrm{LY}_\bullet\mathrm{C}$	type of $\omega^\bullet_\bullet\mathrm{C}$
countable compactum	$\mathrm{LY}_\infty\mathrm{C}$	not $\omega^\infty_2\mathrm{C}$
countable compactum	$\mathrm{LY}_\infty\mathrm{C}$	$\omega^\infty_\infty\mathrm{C}$
perfect compact subset of $\mathbb{R}^3$	$\mathrm{LY}_u\mathrm{C}$	not $\omega^u_2\mathrm{C}$
uncountable compactum	$\mathrm{LY}_2\mathrm{C}$	$\omega^u_\infty\mathrm{C}$

The main result describing connection between LYC and ωC with respect to the cardinality of the scrambled set is formulated in the following theorem.

Theorem B. *Let X be a compact metric space, and let $f \in C(X)$ be $\omega^\infty_2 C$. Then f is LY_2C. In general, any point in an ω^∞-scrambled set of f forms a LY-scrambled set with a suitable point in X.*

5 LY-chaos and the cardinality of the scrambled sets

As it was said in the previous section the implication $\mathrm{LY}_2\mathrm{C} \Rightarrow \mathrm{LY}_u\mathrm{C}$ is not true in general (recall that it is true if the space is an interval or a circle).

We construct two maps on different spaces for which each LY-scrambled set contains exactly two points.

The first one is constructed on a Cantor set and the second one on a set which is a two-dimensional arcwise connected continuum with empty interior endowed with the relative topology of $\mathbb{R}^2$.

Finally, we can propose an open problem: for which spaces the implication $\mathrm{LY}_2\mathrm{C} \Rightarrow \mathrm{LY}_u\mathrm{C}$ is true?

6 Chaos, transitivity and recurrence

In this section we disprove, with foregoing Theorem C, the second part of the main result by L. Wang, Z. Chu and G. Liao in [WChL] that there is an uncountable set $T \subset \Sigma$ with $T \subset R(\sigma) \setminus UR(\sigma)$ such that $\sigma : T \to T$ is uniquely ergodic.

Theorem C. *Let X be a compact metric space and $f \in C(X)$. Then there is no $T \subset R(f) \setminus UR(f)$ such that $f : T \to T$ is uniquely ergodic.*

We also improve the result by M. Babilonová-Štefánková from [BS] and summarize analogous results by M. Babilonová from [Ba2] and M. Lampart from [1].

Theorem D. *Any bitransitive map $f \in C(I)$ is topologically conjugate to a map $g \in C(I)$ which satisfies the following conditions:*

(i) g is extremaly LYC with LY-scrambled set S with full Lebesgue measure and $S \subset R(g) \setminus UR(g)$,

(ii) *g is ωC and every ω-scrambled set Ω has zero Lebesgue measure and $\Omega \subset R(g) \setminus UR(g)$,*

(iii) *g is dC with d-scrambled set D with full Lebesgue measure and $D \subset R(g) \setminus UR(g)$.*

7 Connections between distributional chaos, LY-chaos and ω-chaos

In this section we describe relations between notions of distributional chaos, LY-chaos and ω-chaos.

If the space X is the unit closed interval (or a circle, respectively), then it was proved in [Li] ([Miy], resp.) that ω_u^uC, ω_2^uC and PTE are equivalent (PTE denotes positive topological entropy, for definition see, e.g., [Wa]). Equivalence of PTE and dC was proved in [SS] ([Ma], resp.). PTE implies LY$_u$C, see [BGKM] ([BGKM], resp.) but not conversely [S3] ([Ma], resp.). Finally, equivalence of LY$_u$C and LY$_2$C was discussed in Section 4. In the following diagram 1. we summarize all mentioned implications. A missing arrow means that the implication is not valid, except for implications that follow by transitivity.

diag. 1.
(for an interval or a circle)

$$\begin{array}{ccccc}
\omega_2^u C & & & & \\
\Updownarrow & & & & \\
\omega_u^u C & \Longleftrightarrow & PTE & \Longleftrightarrow & dC \\
& & \Downarrow & & \\
& & LY_u C & & \\
& & \Updownarrow & & \\
& & LY_2 C & &
\end{array}$$

Let us explore the same situation for general compact metric spaces. Let us start with the following theorem.

Theorem E. *There is a perfect compact subspace $\mathbb{X} \subset \mathbb{R}^2$ possessing a continuous map $F : \mathbb{X} \to \mathbb{X}$ which is $\omega_u^u C$ but not dC and has zero topological entropy.*

Proof. By [BRS] (Lemma 4.6.) there is a compact subspace $X \subset \Sigma$ such that σ restricted to X is ω_2^uC but not dC. There is a homeomorphism $h : \Sigma \to C$ (see, e.g., [Fu]), where $C \subset I$ is the Cantor ternary set.

By collapsing $\{0\} \times C$ in $h(X) \times C$ into a point, we get a compact metrizable space. Denote it by $h(X)'C$. One can think about it as a subspace of $\mathbb{R}^2$. The topology on $h(X)'C$ is given by the metric inherited from $\mathbb{R}^2$. We can imagine the space $h(X)'C$ as a union of slices S_i with one common point — zero. (So each S_i is homeomorphic to $h(X)$ and the set $h(X)'C$ is perfect, since each its point is accumulation one.)

Let $\mathbb{X} = h(X)'C$ and $F : \mathbb{X} \to \mathbb{X}$ be such a map that F restricted to S_i is equal to $h(\sigma)$, for each i.

To verify that F is not dC and has zero topological entropy it suffices to use the same arguments as in the proof of Lemma 4.6. in [BRS].

On the other hand, F is ω_u^uC with ω-scrambled set $\Omega = \bigcup_i \{x_i\}$, where x_i is a suitable point from $S_i \setminus \{0\}$. $\square$

Most of above mentioned implications are not valid for general compact metric spaces. There is an example of dC map on a minimal space in [FPS2], hence it is not ω_2^∞C. (Let us note that there is no ω_2^∞C map on minimal system.) By Theorem E the converse implication is not valid. Theorem E gives a counterexample that ω_u^uC does not imply PTE. There are many examples on minimal sets with PTE (see, e.g., [HK]), hence PTE does not imply ω_2^∞C. Validity of implications between ω_2^∞ and LY$_2$C was discussed in Section 4. Counterexamples that no implication between LYC and dC is valid, can be found in [SS] and [Ba1]. It was proved by [BGKM] that PTE implies LY$_u$C, the converse implication does not hold (see diag. 1.). Finally, there is an example of a map which is dC and has zero topological entropy in [BRS] (or [FPS2]). The converse implication is stated as an open problem in [BRS] with suspected positive answer. For completeness we give all these implications in the next diagram 2. and again a missing arrow means that the implication is not valid, except for implications that follow by transitivity.

diag. 2.
(for general compact metric space)

$$\begin{array}{ccccc}
\omega_u^u C & & & & \\
\Downarrow & & & & \\
\omega_2^\infty C & \Longrightarrow & LY_2 C & & \\
 & & \Uparrow & & \\
 & & LY_u C & & \\
 & & \Uparrow & & \\
 & & PTE & \stackrel{?}{\Longrightarrow} & dC
\end{array}$$

8 Publications concerning the Thesis

[1] M. Lampart, *Scrambled sets for transitive maps.* Real Anal. Exch. 27 (2001-02), no. 2, 801-808.

[2] M. Lampart, *Two kinds of chaos and relations between them.* Acta Math. Univ. Comen. 72 (2003), no. 1, 119-129.

[3] J. L. García Guirao and M. Lampart, *Li and Yorke chaos with respect to the cardinality of the scrambled sets.* Chaos, Solitons and Fractals 24 (2005), 1203 - 1206.

[4] M. Lampart, *Chaos, transitivity and recurrence.* Grazer Math. Ber. — submitted

9 Quotations by other authors

[5] J. Smítal, *Various notions of chaos, recent results, open problems*, Real Anal. Exch. Summer Symposium 2002, 81 – 86. (cf. [2])

[6] J. L. García Guirao, *Omega-limit sets and topological entropy for twodimensional triangular maps*, Ph.D. thesis, University of Murcia, Spain, 2004. (cf. [2])

10 Presentations

[7] The 29^{th} Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech Republic, January 2001. Talk on: *"Small and big scrambled set for transitive maps"*.

[8] 5^{th} Czech - Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 18 - 23, 2001. Talk on: *"Scrambled sets for transitive maps"*.

[9] The 30^{th} Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech republic, January 2002. Talk on: *"Scrambled sets for transitive maps"*.

[10] 6^{th} Czech-Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, June 9 - 16, 2002. Talk on: *"Two kinds of chaos and relations between them"*.

[11] International Mathematics Students Competition (SVOČ), Prague, May 13 - 15, 2002. *"Two kinds of chaos and relations between them"*. The paper wons the 2nd price.

[12] The 31^{th} Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech republic, January 2003. Tolk on: *"Two kinds of chaos and relations between them"*.

[13] Summer Symposium in Real Analysis XXVII, Opava, Czech Republic, June 23 - 29, 2003. Talk on: *"Two kinds of chaos and relations between them"*.

[14] Universidad Politecnica de Cartagena, Spain, November 18, 2003. During working visit [19]. Lecture on: *"Two kinds of chaos and relations between them"*.

[15] The 32^{nd} Winter School on Abstract Analysis - Section Topology, Doksy, January 31 - February 7, 2004. Talk on: *"Li and Yorke chaos with respect to the cardinality of the scrambled sets"*.

[16] European Conference on Iteration Theory (ECIT' 04), Batschuns, Austria. Talk on: *"Chaos, transitivity and recurrence"*.

[17] 8^{th} Czech - Slovak Workshop on Discrete Dynamical Systems, Praděd, Czech Republic, September 18 - 25, 2004. Talk on: *"Chaos, transitivity and recurrence"*.

11 Working visits

[18] Summer School on Dynamical Systems, University of Porto, Portugal, June 30 - July 4 2003.

[19] One Trimester at University of Murcia, Murcia, Spain, September - December, 2003.

References

[Ba1] M. Babilonová: *Distributional chaos for triangular maps.* Ann. Math. Silesianae, 13 (1999), 33-38.

[Ba2] M. Babilonová: *The bitransitive continuous maps of the interval are conjugate to maps extremely chaotic a.e.* Acta Math. Univ. Come. LXIX, 2 (2000), 229-232.

[BS] M. Babilonová-Štefánková: *Extreme chaos and transitivity.* Int. Jour. Bif. Chaos, 7 (2003), 1695-1700.

[BGKM] F. Blanchard, E. Glasner, S. Kolyada and A. Maass: *On Li-Yorke pairs.* J. Reine Angew. Math. 547 (2002), 51-68.

[Bl] A. M. Blokh: *The "spectral" decomposition for one-dimensional maps.* Dynam. Reported. 4 (1995), 1-59.

[BH] A. M. Bruckner and T. Hu: *On scrambled sets for chaotic functions.* Trans. Amer. Math. Soc. 301 (1987), 289-297.

[BRS] F. Balibrea, L. Reich and J. Smítal, *Iteration Theory: Dynamical Systems and functional equations.* Internat. J. Bifur. Chaos 13 (2003), 1627 - 1647.

[FPS1] G. L. Forti, L. Paganoni and J. Smítal: *Strange triangular maps of the square*, Bulletin Austral. Math. Soc. 51 (1995), 395-415.

[FPS2] G. L. Forti, L. Paganoni and J. Smítal: *Dynamics of momeomorphisms on minimal sets generated by triangular mappings*, Bulletin Austral. Math. Soc. 59 (1999), 1-20.

[Fu] H. Furstenberg: *Recurrence in Ergodic Theory and Combinational Number Theory.* Princeton University Press, Princeton, New Jersey, 1981.

[HK] F. Hahn and Y. Katznelson: *On the entropy of uniquely ergodic transformations.* Trans. Amer. Math. Soc. 126 (1967), 335–360.

[HY] W. Huang and X. Ye: *Homeomorphisms with the whole compacta being scrambled sets*, Ergod. Th. & Dynam. Sys. 21 (2001), 77-91.

[Ka] I. Kan: *A chaotic functions possessing a scrambled set with positive Lebesgue measure.* Proc. Amer. Math. Soc 92 (1984), 45-49.

[Ku] M. Kuchta: *Characterization of chaos for continuous maps of the circle.* Proc. Europ. Conf. on Iteration Theory, Spain 1987, World Scientific, Singapore (1989), 427-430.

[KS] M. Kuchta and J. Smítal: *Two point scrambled set implies chaos.* Real Anal. Exch. 27 (2001/02), no. 2, 801-808.

[La] M. Lampart: *Scrambled sets for transitive maps.* Real Anal. Exch. 27 (2001/02), no. 2, 801-808.

[Li] S. Li: *ω-chaos and topological entropy.* Trans. Amer. Math. Soc. 339 (1993), 243-249.

[LY] T. Y. Li and J. Yorke: *Period three implies chaos.* Amer. Math. Monthly 82 (1975), 985-992.

[Ma] M. Málek: *Distributional chaos and spectral decomposition of dynamical systems on the circle.* Topology Appl. 135 (2004), 215 - 229. ISSN 0166-8641.

[Mi] M. Misiurewicz: *Chaos almost everywhere.* Iteration Theory and its Functional Equations, Lecture Notes in Math. 1163, Springer, Berlin 1985, 125-130.

[Miy] M. Miyazawa: *Chaos and entropy for circle maps*. Tokyo J. Math. 25 (2002), no. 2, 453-458.

[SS] B. Schweizer and J. Smítal: *Measures of chaos and a spectral decomposition of dynamical systems on the interval*. Trans. Amer. Math. Soc. 344 (1994), 737-754.

[S1] J. Smítal: *A chaotic functions with some extremal properties*. Proc. Amer. Math. Soc 87 (1983), 54-56.

[S2] J. Smítal: *A chaotic functions with a scrambled set of positive Lebesgue measure*. Proc. Amer. Math. Soc 92 (1984), 50-54.

[S3] J. Smítal: *Chaotic functions with zero topological entropy*. Trans. Amer. Math. Soc. 297 (1986), no. 1, 269-282.

[Wa] P. Walters: *An introduction to ergodic theory*. Springer, New York, 1982.

[WChL] L. Wang, Z. Chu and G. Liao: *Recurrence and unique ergodicity*. Top. and its App. 138 (2004), 97-107.