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Irregular recurrence in compact metric spaces

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Contents

1. Introduction

This thesis is based on three independent papers connected by one common subject - Irregular Recurrence (e.g. properties of systems possessing at least on irregularly recurrent point). We attach these papers as a supplement.

First paper "Distributional chaos and irregular recurrence" is joint work with my supervisor, Prof. Jaroslav Smítal and was published in Nonlinear Analysis in 2010 (see ref. [1]). Second paper entitled "Counterexamples to the open problem by Zhou and Feng on minimal center of attraction" is also joint work with my supervisor, Prof. Jaroslav Smítal and was published in Nonlinearity in 2012 (see ref. [2]). The last paper "Irregular recurrence in compact metric spaces" was submitted to Archiv der Mathematik in 2012 (see ref. [3]).

Properties of irregularly recurrent points were studied by Chinese mathematician Z. Zhou et. al. in the nineties. These points also have close connection to the topological entropy, and distributional chaos. There are lot of open problems concerning these points, see [9] or [10]. In papers [1], [2] and [3] there are solutions to some of these problems, but there are also ones which still remain open.

Paper [1] studies some relations between distributional chaos and existence of an irregularly recurrent point. The main result shows that existence of an irregularly recurrent point does not imply the strongest version of distributional chaos, DC1, even in the class of triangular maps.

Paper [2] answers one of the problems in [9], in particular gives counterexamples which show that there is no connection between existence of an irregularly recurrent point and existence of an invariant measure with support equal to the center of attraction of this point.

Paper [3] gives some properties of irregularly recurrent points and generalizes counterexamples from [2] to systems with positive topological entropy, which solves a problem from [10].

2. Basic terminology and notation

In this section we describe basic terminology and notation, common for all three papers. We recall other definitions, used in specific problems, in the corresponding sections.

We work with a compact metric space X with metric d and a continuous map f from X to itself. By $f^n(x)$, where n is a nonnegative integer, we denote the n-th iteration of x under f. The sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$, is the *forward trajectory* of x under f. By $\omega_f(x)$ we denote the *ω-limit set* of x, which is the set of all cluster points of trajectory of x. This ω-limit set is maximal, when it is contained in no larger ω-limit set. By $\omega(f)$ we mean the union of all ω-limit sets of f. *Minimal set* of f is non-empty closed set M, such that $f(M) = M$ and no proper subset of M has these properties.

When defining an irregularly recurrent point, we start with the original definitions from [9]:

Definition 1. A point $x \in X$ is *weakly almost periodic* if for $\forall \varepsilon > 0 \;\; \exists N > 0$ such that

$$\sum_{i=0}^{nN-1} \chi_{B(x,\varepsilon)}(f^i(x)) \geq n, \quad \forall n > 0, \tag{1}$$

where $\chi_{B(x,\varepsilon)}$ denotes the characteristic function of the set $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Definition 2. A point $x \in X$ is *quasi-weakly almost periodic* if for $\forall \varepsilon > 0 \ \ \exists N > 0$ and an increasing sequence of positive integers $\{n_j\}$ such that

$$\sum_{i=0}^{n_j N-1} \chi_{B(x,\varepsilon)}(f^i(x)) \geq n_j, \quad \forall j > 0. \tag{2}$$

The set of all weakly almost periodic points of a map f is denoted by $W(f)$ and the set of all quasi-weakly almost periodic points is denoted by $QW(f)$. Clearly, by (1) and (2), $W(f) \subset QW(f)$. The notion of sets $QW(f)$ and $W(f)$ were introduced by Z.Zhou in 1993 (see [8]) to investigate the structure of measure centre. Some of the problems from [9] and [10] consider points which are quasi-weakly almost periodic, but not weakly almost periodic. For simplicity, we started to call such points *irregularly recurrent* and we denote the set of all irregularly recurrent points of f by $IR(f)$, e.g. $IR(f) = QW(f) \setminus W(f)$.

In [1] - [3] we used a more convenient definition of irregularly recurrent points. For $x \in X$ and $t > 0$, let

$$\Psi_x(f,t) = \liminf_{n\to\infty} \tfrac{1}{n}\#\{0 \leq j < n; d(x, f^j(x)) < t\}, \tag{3}$$

$$\Psi_x^*(f,t) = \limsup_{n\to\infty} \tfrac{1}{n}\#\{0 \leq j < n; d(x, f^j(x)) < t\}. \tag{4}$$

Thus, $\Psi_x(f,t)$ and $\Psi_x^*(f,t)$ are the *lower* and *upper Banach density* of the set $\{n \in \mathbb{N}; f^n(x) \in B(x,t)\}$, respectively. Then we define irregularly recurrent point as follows:

Definition 3. Point x is quasi-weakly almost periodic, if $\Psi_x^*(f,t) > 0$ for every t and is weakly almost periodic, if $\Psi_x(f,t) > 0$ for every t.
Point x is irregularly recurrent, if $\Psi_x^*(f,t) > 0$ for all t and $\Psi_x(f,t) = 0$ for at least one t.

The equivalence between original and new definition is obvious, the proof is given in [4].

We denote by $R(f)$ the set of all recurrent points of f, and by $UR(f)$ the set of all uniformly recurrent points of f. Then, by definitions,

$$UR(f) \subseteq W(f) \subseteq QW(f) \subseteq R(f) \subseteq \omega(f). \tag{5}$$

At last, we should recall the notations of triangular map and topological entropy. *Skew-product* map $X \times Y \longrightarrow X \times Y$ is a map $F : (x,y) \mapsto (f(x), g(x,y))$ continuous with respect to the max-metric on $X \times Y$. In the particular case when $X = Y$ is the unit interval $I = [0,1]$, F is a *triangular map.*

A set $A \subset X$ is (n,ε)-*separated* if, for any distinct points $x_1, x_2 \in A$, there is i $(0 \leq i < n)$ such that $d(f^i(x_1), f^i(x_2)) > \varepsilon$. For $Y \subset X$, denote by $s_n(\varepsilon, Y, f)$ the maximum possible number of points in an (n,ε)- separated subset of Y. Let

$$h(f|_Y) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, Y, f). \tag{6}$$

The *topological entropy* of the map f is defined as $h(f) = h(f|_X)$.

3. Some properties of irregularly recurrent points

This section is based on results from [3].

Properties of quasi-weakly almost periodic points and weakly almost periodic points were studied in some papers by Z. Zhou and other authors, while properties of irregularly recurrent points were studied almost nowhere, although there are many open problems concerning these points, which still remain open.

Theorem 1. (Cf. [3]). *If $f \in C(X)$, then*

i) $f(QW(f)) = QW(f)$,

ii) $f(W(f)) = W(f)$.

Corollary 1. (Cf. [3]). $f(IR(f)) = IR(f)$.

Before stating next property, let us recall some terminology. Let X with $f : X \to X$ and Y with $g : Y \to Y$ be a compact metric spaces and continuous maps on them and let $\varphi : X \to Y$ be continuous and surjective map such that $\varphi \circ f = g \circ \varphi$. Then (Y, g) is called a *factor* of a system (X, f), and map φ is the corresponding *factor map*.

Theorem 2. (Cf. [3]). *Let (Y, g) be a factor of (X, f), via factor map φ. Then $\varphi(QW(f)) = QW(g)$, $\varphi(W(f)) = W(g)$ and subsequently $\varphi(IR(f)) = IR(g)$.*

Part i) of the next theorem is proved in [8], but we can give simpler argument and extend it to the part ii) of the theorem.

Theorem 3. (Cf. [3]). *If $f \in C(X)$ and $m \in \mathbb{N}$, then*

i) $W(f) = W(f^m)$,

ii) $QW(f) = QW(f^m)$.

Corollary 2. (Cf. [3]). $IR(f) = IR(f^m)$

4. Distributional chaos and irregular recurrence

This section is based on results from [1].

Two points $x, y \in X$ are *proximal*, if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and they are *asymptotic*, if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. Points x, y form a *Li-Yorke pair*, if they are proximal, but not asymptotic. A system (X, f) is *Li-Yorke chaotic*, briefly *LYC*, if X contains at least one Li-Yorke pair.

For any pair $x, y \in X$ and any $0 < t \leq diam(X)$ let

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n; d(f^j(x), f^j(y)) < t\}, \tag{7}$$

$$\Phi^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n; d(f^j(x), f^j(y)) < t\}. \tag{8}$$

We call Φ_{xy} and Φ^*_{xy} the *lower* and *upper distribution functions* of x, y, respectively. Obviously, $\Phi_{xy}(t) \leq \Phi^*_{xy}(t)$ for any $0 < t \leq diam(X)$. If $\Phi_{xy}(t) < \Phi^*_{xy}(t)$ for all t in an interval, we write $\Phi_{xy} < \Phi^*_{xy}$. There are three types of *distributional chaos*: $DC1$, $DC2$ and $DC3$. The conditions for points x, y to form a *distributionally chaotic pair* of type 1,2 or 3 are following:

(DC1) $\Phi^*_{xy} \equiv 1$ and $\Phi_{xy}(t) = 0$ for some $t > 0$,

(DC2) $\Phi^*_{xy} \equiv 1$ and $\Phi_{xy} < \Phi^*_{xy}$,

(DC3) $\Phi_{xy} < \Phi^*_{xy}$.

We call map f distributionally chaotic of type 1,2 or 3, if there is at least one distributionally chaotic pair of type 1,2 or 3 in X, respectively. Straight from the definitions we can see that $DC1 \Longrightarrow DC2 \Longrightarrow DC3$. It is also known and easy to see that $DC2 \Longrightarrow LYC$. $DC2$ and $DC3$ are also implied by positive topological entropy (see [21]), while $DC1$ not (see [15]). The strongest type of chaos, $DC1$, was originally introduced in [17], $DC2$ and $DC3$ are its generalizations, see [15] or [16].

Since the definition of an irregularly recurrent point is fairly similar to the definitions of distribution functions, we consider problem whether the existence of an irregularly recurrent point is somehow related to chaos of any type.

For continuous maps on the interval all these properties (positive topological entropy, all three types of distributional chaos and the existence of irregularly recurrent point) are equivalent. Equivalence between positive topological entropy and $IR(f) \neq \emptyset$ is proved in [4]. This solves an open problem from [9]. This result can be also extended for some more general compact metric spaces, like topological graphs or trees, but not for all compact metric spaces.

Now we put the results for continuous maps on general compact metric space. First result is that irregular recurrence implies LYC.

Theorem 4. (Cf. [1]). *Let f be a continuous map of a compact metric space X such that $IR(f) \neq \emptyset$. Then f is LYC.*

For stronger types of chaos the situation is not that easy.

Theorem 5. (Cf. [1]). *There is a skew-product map f of the space $C \times I$, where C is the Cantor set such that*

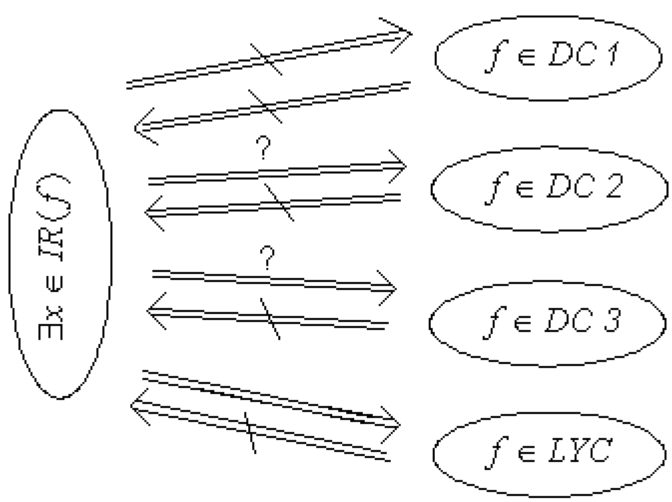
i) $IR(f) \neq \emptyset$,
ii) *f is $DC2$ but not $DC1$,*
iii) *f has zero topological entropy.*

On the other hand we can show that $DC1$ does not imply existence of an irregularly recurrent point. Moreover, both counterexamples can be found in the class of triangular maps.

Theorem 6. (Cf. [1]). *For triangular maps, the properties $DC1$ and $IR(f) \neq \emptyset$ are independent, i.e. there is no implication between them.*

This result also contributes to the problem of classification of triangular maps, which has been formulated in the eighties by A. N. Sharkovsky (very recently, in 2012, the remaining 3 open problems by Sharkovsky were solved, see [22], [23], [24]).

There still remain two open problems, whether irregular recurrence implies $DC2$, or at least $DC3$. The whole situation is obvious from the following graph:



Based on [21], there is following conjecture:

Conjecture 1. Using similar tool as in [21], it can be proved that existence of an irregularly recurrent point implies chaos $DC2$ (and subsequently $DC3$).

5. Irregular recurrence, invariant measures and topological entropy

This section is based on results from [2] and [3].

Let $M(X,f)$ be the set of *invariant probability measures* of f on X. Measure μ is probability, if $\mu(X)=1$, and $\mu(\emptyset)=0$. Measure μ is invariant if for every measurable $A\subseteq X$, $\mu(f^{-1}(A))=\mu(A)$. By S_μ we denote the *support* of measure μ, i.e. the minimal closed set of a full measure (of μ-measure 1 in the case of probability measures). Let $M_x(f)$ be the set consisting of the limit points of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$, where δ_x is the atomic probability measure on X with support $\{x\}$. When speaking on convergence of sequences of measures we always consider the classical pointwise convergence, so that $\mu_k\to\mu$ means $\mu_k(B)\to\mu(B)$, for every open (or equivalently, Borel) set B. It is well-known that $M_x(f)\subseteq M(X,f)$. A set $E\subseteq X$ is called the *minimal centre of attraction* of a point $x\in X$ if E is the minimal closed set such that $f(E)\subseteq E$ and, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\#\{i<n; f^i(x)\in B(E,\varepsilon)\}=1, \tag{9}$$

where $B(E,\varepsilon)$ denotes the open ε- neighbourhood of the set E (as in Section 2). We denote the minimal centre of attraction of a point $x\in X$ by $C_x(f)$.

Remark 1. (see [9]) It is known that for any $x\in IR(f)$ there is a $\mu\in M_x(f)$ such that $S_\mu\neq C_x(f)$ and $M_x(f)$ is not a singleton. On the other hand, if $x\in R(f)\setminus QW(f)$, then $S_\mu\neq C_x(f)$ for every $\mu\in M_x(f)$ and if $x\in W(f)$, then $S_\mu=C_x(f)$ for every $\mu\in M_x(f)$.

Since $W(f)\subseteq QW(f)\subseteq R(f)$, it is natural to ask the following:

Problem 1. ([9], Open problem 4 in the second series) *Let $x\in IR(f)$. Is there a $\mu\in M_x(f)$ such that $S_\mu=C_x(f)$?*

For quasi-weakly almost periodic points, $C_x(f)=\omega_f(x)$ (see [9]). Since we consider the point x to be irregularly recurrent (and $IR(f)\subset QW(f)$), we can re-formulate the problem in the following way:

Problem 2. *Let $x\in IR(f)$. Is there a $\mu\in M_x(f)$ such that $S_\mu=\omega_f(x)$?*

The main result from [2] which helps to solve the posed problem is the following theorem, which gives a characterization of points in $IR(f)$ with the property that $\omega_f(x)=S_\mu$.

Theorem 7. (Cf. [2]). *Let (X,f) be a topological dynamical system where X is a compact metric space and $z\in IR(f)$. Then there is a $\mu\in M_z(f)$ with support $S_\mu=\omega_f(z)$ if and only if there is a sequence of positive integers $m_1<m_2<\ldots$ such that, for every neighbourhood G of z,*

$$\liminf_{k\to\infty}\frac{1}{m_k}\#\{0\le j<m_k; f^j(z)\in G\}>0. \tag{10}$$

If this is the case, then every limit point of the sequence $\{\mu_k\}$, where $\mu_k=\frac{1}{m_k}\sum_{n=0}^{m_k-1}\delta_{f^n(z)}$ is a measure $\mu\in M(X,f)$ with support $\omega_f(z)$.

By Remark 1, the Theorem is true also if $IR(f)$ is replaced by $R(f)$, i.e. it characterizes recurrent points with the property that $\omega_f(x)=S_\mu$ for som $\mu\in M_x(f)$. We can also show that this classification is non-trivial.

Theorem 8. (Cf. [2]).

i) *There is a* $z_1 \in \Sigma_2$ *such that* $z_1 \in QW(\sigma)$ *and* $\omega_\sigma(z_1)$ *is the support of no invariant measure* $\mu \in M(\Sigma_2, \sigma)$.

ii) *There is a* $z_2 \in \Sigma_2$ *such that* $z_2 \in IR(\sigma)$ *and* $\omega_\sigma(z_2)$ *is the support* S_μ *of an invariant measure* $\mu \in M(\Sigma_2, \sigma)$.

We construct both examples in the space $\Sigma_2 = \{0,1\}^{\mathbb{N}}$, the space of all infinite sequences of zeros and ones, with standard one-sided shift map σ. We equip this space with a metric ρ of pointwise convergence, e.g., $\rho(x,y) = \frac{1}{k}$, where k is the first coordinate where x and y are different. For showing that there is a system with irregularly recurrent point which omega-limit set is support of an invariant measure $\mu \in M_x(f)$, we could also use skew-product map $F : Q \times I \to Q \times I$, where Q is the Cantor set from [1]. The result would follow by Theorem 7.

Subshifts constructed in [2] (and mentioned in Theorem 8) can be also used to solve another open problem from [10], whether every subshift of (Σ_2, σ) possessing an irregularly recurrent point must have positive topological entropy. The answer to this problem is negative. In [2] there is proved that points z_1 and z_2 are irregularly recurrent, so it is enough to show that topological entropy of maps $\sigma|_{\omega_\sigma(z_1)}$ and $\sigma|_{\omega_\sigma(z_2)}$ is zero. We can do this easily by using the formula

$$h(\sigma|_{\omega_\sigma(x)}) = \limsup_{n\to\infty} \frac{1}{n} \log P_n, \tag{11}$$

where P_n denotes the number of different blocks of zeros and ones of the length n in x (see e.g. [14]).

Moreover, the subshifts from Theorem 8 can be slightly changed such that all its properties will be preserved (it means changed points z_1 and z_2 will still be irregularly recurrent, for z_1 there will still be no invariant measure $\mu \in M_{z_1}(f)$ such that $S_\mu = \omega_\sigma(z_1)$ and for z_2 there will still be an invariant measure with this property), but topological entropy of shifts restricted to ω-limit sets of these points will be positive.

Theorem 9. (Cf. [3]). *There are irregularly recurrent points* $z_1, z_2 \in \Sigma_2$ *such that* $\omega_\sigma(z_1)$ *is and* $\omega_\sigma(z_2)$ *is not the support of an invariant measure* $\mu \in M(\Sigma_2, \sigma)$, *but* $h(\sigma|_{\omega_\sigma(z_1)}) = h(\sigma|_{\omega_\sigma(z_2)}) = 0$.

Theorem 10. (Cf. [3]). *There are irregularly recurrent points* $w_1, w_2 \in \Sigma_2$ *such that* $\omega_\sigma(w_1)$ *is and* $\omega_\sigma(w_2)$ *is not the support of an invariant measure* $\mu \in M(\Sigma_2, \sigma)$, *but* $h(\sigma|_{\omega_\sigma(w_1)}) > 0, h(\sigma|_{\omega_\sigma(w_2)}) > 0$.

Theorem 9 gives the negative answer to the open problem and Theorems 9 and 10 together show that there is no connection between positive topological entropy, irregular recurrence and existence of an invariant measure, which support is equal to omega-limit set of the irregularly recurrent point generating this subshift. The fact that irregular recurrence does not imply positive topological entropy is showed in Section 4.

6. Talks at conferences

- Summer Symposium in Real Analysis XXXII, Chicago, Illinois, USA, June, 2008
  Talk: Continuous maps of the interval and of the square disproving conjectures on Hausdorff dimension and invariant measures
- 12th Czech-Slovak Workshop on Discrete Dynamical Systems, Pustevny, Czech Republic, September, 2008
  Talk: Solution of a problem by Zhou and Feng concerning invariant measures
- Summer Symposium in Real Analysis XXXIII, Durant, Oklahoma, USA, June, 2009
  Talk: Distributional chaos and irregular recurrence

- 13th Czech-Slovak Workshop on Discrete Dynamical Systems, Jeseníky, Czech Republic, September, 2009
  Talk: Distributional chaos and irregular recurrence
- Summer Symposium in Real Analysis XXXIV, Wooster, Ohio, USA, July, 2010
  Talk: Irregular recurrence in compact metric spaces
- 14th Czech-Slovak-Spanish Workshop on Discrete Dynamical Systems, La Manga del Mar Menor, Spain, September, 2010
  Talk: Irregular recurrence in compact metric spaces
- Summer Symposium in Real Analysis XXXV, Budapest, Hungary, June, 2011
  Talk: Topological entropy and irregular recurrence
- 15th Czech-Slovak Workshop on discrete dynamical systems, Banská Bystrica, Slovakia, June, 2011
  Talk: Topological entropy and irregular recurrence
- International Conference on Numerical Analysis and Applied Mathematics 2011, Halkidiki, Greece, September, 2011
  Talk: Irregular recurrence in compact metric spaces
- Summer Symposium in Real Analysis XXXVI, Berks, Pennsylvania, USA, July, 2012
  Talk: Counterexamples to the open problem by Zhou and Feng on minimal center of attraction
- European Conference on Iteration Theory, Ponta Delgada, Azores, Portugal, September, 2012
  Talk: Counterexamples to the open problem by Zhou and Feng on minimal center of attraction

7. Publications concerning the Thesis

[1] Obadalová L, Smítal J. Distributional chaos and irregular recurrence, Nonlin Anal A - Theor Meth Appl 72 (2010), 2190 - 2194. (IF 1.5)
[2] Obadalová L, Smítal J. Counterexamples to the open problem by Zhou and Feng on minimal center of attraction, Nonlinearity 25 (2012), 1443 - 1449. (IF 1.4)
[3] Obadalová L, Irregular recurrence in compact metric spaces (submitted to Archiv der Mathematik in august 2012).

8. Other related publications (not used in the Thesis)

[4] Obadalová L, Topological entropy and irregular recurrence (accepted in Analysis in Theory and Applications in november 2012).
[5] Obadalová L, Continuous maps of the interval and of the square disproving conjectures on Hausdorff dimension and invariant measures, Real Analysis Exchange, Summer Symposium 2008, (2009), 125130. ISSN 0147-1937 (USA)
[6] Obadalová L, Distributional chaos and irregular recurrence, Real Analysis Exchange, Summer Symposium 2009 (2010), 6166.
[7] Obadalová L, Irregular recurrence in compact metric spaces, Real Analysis Exchange, Summer Symposium 2010 (2011), 1922.

References

[1] Obadalová L, Smítal J. Distributional chaos and irregular recurrence, Nonlin Anal A - Theor Meth Appl **72** (2010), 2190 – 2194.

[2] Obadalová L, Smítal J. Counterexamples to the open problem by Zhou and Feng on minimal center of attraction, Nonlinearity **25** (2012), 1443 – 1449.

[3] Obadalová L. Irregular recurrence in compact metric spaces (submitted in august 2012).

[4] Obadalová L. Topological entropy and irregular recurrence, Analysis in Theory and Applications, accepted in november 2012.

[5] Obadalová L. Continuous maps of the interval and of the square disproving conjectures on Hausdorff dimension and invariant measures, Real Analysis Exchange, Summer Symposium 2008, (2009), 12 –13. ISSN 0147-1937 (USA)

[6] Obadalová L. Distributional chaos and irregular recurrence, Real Analysis Exchange, Summer Symposium 2009 (2010), 61 – 66.

[7] Obadalová L. Irregular recurrence in compact metric spaces, Real Analysis Exchange, Summer Symposium 2010 (2011), 19 – 22.

[8] Zhou Z. Weakly almost periodic point and measure centre. Science in China (Ser. A) **36** (1993) 142 – 153.

[9] Zhou Z, Feng L. Twelve open problems on the exact value of the Hausdorff measure and on topological entropy. Nonlinearity **17** (2004), 493 – 502.

[10] Zhou Z. and Feng L. Some problems on fractal geometry and topological dynamical systems. Anal. Theor. Appl. **25** (2009), 5 – 15.

[11] Dai X, Zhou Z, Geng X. Some relations between Hausdorff dimensions and entropies, Science in China (Series A), **41** (1998), 1068 – 1075.

[12] Block L.S, Coppel W.A. Dynamics in One Dimension, Springer-Verlag, Berlin Heidelberg, 1992

[13] Walters P. An Introduction to Ergodic Theory, Springer, New York, 1982.

[14] Denker M, Grillenberger Ch, Sigmund K. Ergodic Theory on Compact Spaces, Springer, New York, 1976.

[15] Smítal J, Štefánková M. Distributional chaos for triangular maps, Chaos, Solitons and Fractals **21** (2004), 1125–1128.

[16] Balibrea F, Smítal J, Štefánková M. The three versions of distributional chaos. Chaos, Solitons and Fractals **23** (2005), 1581 – 1583.

[17] Schweizer B and Smítal J. Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. **344** (1994), 737 – 854.

[18] Paganoni L, Smítal J. Strange distributionally chaotic triangular maps. Chaos, Solitons and Fractals **26** (2005), 581 – 589.

[19] Paganoni L, Smítal J. Strange distributionally chaotic triangular maps II. Chaos, Solitons and Fractals **28** (2006), 1356 – 1365.

[20] Paganoni L, Smítal J. Strange distributionally chaotic triangular maps III. Chaos, Solitons and Fractals **37** (2008), 517 – 524.

[21] Downarowicz T. Positive topological entropy implies chaos DC2, Proc. AMS (to appear), 2012.

[22] Downarowicz T. Minimal subsystems of triangular maps of type 2^∞, preprint, 2012.

[23] Downarowicz T. and Štefánková M. Embedding Toeplitz systems in triangular maps; The last but one problem of the Sharkovsky classification program, Chaos, Solitons and Fractals **45** (2012), 1566 – 1572.

[24] Štefánková M. Strange chaotic triangular maps, Chaos, Solitons and Fractals **45** (2012), 1188 – 1191.