REPORT ON THE HABILITATION THESIS BY DR. HONG VAN LE
"TOPOLOGY OF COMPACT SYMPLECTIC MANIFOLDS"

The theses by Dr. Hong Van Le consists of 6 papers. First two papers deal with famous Arnold conjecture that the number $n(\varphi)$ of isolated fixed points of an exact symplectic transformation φ of a compact symplectic manifold (M, ω) does not exceed the sum of its Betti numbers. Using Floer cohomology, Dr. Hong Van Le estimates $n(\varphi)$ from above by the cup-length in Z_2-cohomology under an assumption that

$$c_1(M)|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}$$

where $c_1(M)$ is the first Chern class of an almost complex structure compatible with ω (s.t. $g = \omega \circ J$ is a Riemannian metric).

In the case, when φ is a non exact symplectic transformation isotopic to the identity, Arnold conjecture is wrong. Dr. Hong Van Le (in joint paper with K. Ono) proposed to estimate $n(\varphi)$ in terms of Novikov $\mathbb{Z}_2$-homology (which are generalization of Morse homology to the case of a closed, but not exact 1-form). She proved that under some conditions, the number $n(\varphi)$ of fixed points of φ is estimated by the sum of Betti numbers of the Novikov cohomology.

In the third paper, Dr. Hong Van Le proposed a generalization of the Gromov-Witten invariant to the case of a symplectic bundle that is a bundle $E \to B$ with fibres diffeomorphic to a symplectic manifold (M, ω) with transition function which take value in symplectic group $Symp(M, \omega)$. She describes properties of these parametrized Gromov-Witten invariants and applies them for investigation of the homotopy group $\pi_1(Symp(M))$. More precisely, she proves that this group is non trivial for some compact symplectic manifolds M.

Fourth paper is devoted to the investigation of the following important problem: Given the homotopy class $[J]$ of an almost complex structure on a 4-dimensional manifold M. When there exists a symplectic form ω compatible with an almost complex structure J from this class (s.t. $\omega \circ J > 0$). Using Yang-Mills theory,
S. Donaldson had constructed an almost complex structure on the K3 surface which has no compatible with $[J]$ symplectic structure. He also defined a free involution p on the set $\{[J]\}$ of homotopic classes of almost complex structures on a compact oriented 4-manifold M^4.
Let M^4 be an oriented compact complex manifold with $b_2^+ > 2$ or $b_2^+ = 1$ and $b_1 = 0$. Using Seiberg -Witten theory, Dr. Hong Van Le had proved that if an almost complex structure J admits a compatible symplectic form , then the class $[p(J)]$ does not admits compatible symplectic form.

The fifth paper deals with a symplectic version of the classical R. Thom result about realization on an appropriate multiple $[N\alpha]$ of a homological class $\alpha \in H_k(M, \mathbb{Z})$ of a compact manifold M by a smooth submanifold.
Let (M, ω) be a compact symplectic manifold and $\alpha \in H_{2k}(M, \mathbb{Z})$ a homological class. Then there is a natural number N such that $[N\alpha] = [S_1] - [S_2]$ is realized as the difference of two symplectic $2k$-manifolds. This is a generalization of a result by S. Donaldson and Auroux.

In the last paper, Dr. Hong Van Le proves that for any complex vector bundle $E \to M$ of rank k over m-dimensional manifold M with Chern classes $c_i \in H^{2i}(M, \mathbb{Z})$ the classes $p \cdot \ell_i \cdot c_i$ are characteristic classes of some complex vector bundle $E' \to M$. This result implies the Thom theorem and its symplectic version, considered in the previous paper.

The thesis contains many deep and interesting results on geometry and topology of compact symplectic manifolds. The author is internationally known expert in symplectic geometry. There is no doubt, that she deserves the rank of "Docent" in the area of Mathematics - Geometry and Global Analysis.

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