MATEMATICKÉ BESEDY

12. 10. 2012

FUNKCIONÁLNE ROVNICE

Nájdite všetky f také, že:

1. $f: \langle 0,\infty) \to \langle 0,\infty)$

a) $f(xf(y))f(y) = f\left(\frac{xy}{x+y}\right)$

b) $f(1) = 0$

c) $f(x) > 0$ pre ľubovoľné $x > 1$

Riešenie:

$y = 1$ $\qquad f(xf(1))f(1) = f\left(\frac{x}{x+1}\right)$

$f(1) = 0 \Rightarrow$

$$\boxed{f\left(\frac{x}{x+1}\right) = 0} \quad \forall x \geq 0$$

Aké sú to čísla $\frac{x}{x+1}$? $\in \langle 0,1)$

$\forall t \in \langle 0,1) \qquad t = \frac{x}{x+1} \Rightarrow x = \frac{t}{1-t}$

$\Rightarrow f(x) = 0 \iff x \in \langle 0,1)$

$f(x) > 0$ pre $x > 1$

Podme sa pozrieť, kedy

$$0 \leq \frac{xy}{x+y} \leq 1 \quad \text{potom } f\left(\frac{xy}{x+y}\right) = 0$$

$$\Updownarrow$$

$$xy \leq x+y$$

$$\cancel{x} \quad x(y-1) \leq y \qquad \text{uvážujme } \underline{y > 1}$$

$$x \in \left\langle 0, \frac{y}{y-1} \right\rangle$$

potom i $f(x f(y)) f(y) = 0$

$$\Updownarrow \qquad \neq 0$$

$$f(x f(y)) = 0 \iff x f(y) \leq 1$$

$$x \leq \frac{1}{f(y)} \quad x \in \left\langle 0, \frac{1}{f(y)} \right\rangle$$

Tie intervaly sa musia rovnať, preto:

$$\frac{1}{f(y)} = \frac{y}{y-1} \iff f(y) = \frac{y-1}{y}$$

a máme $f(y)$ pre $y > 1$.

Riešenie:
$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ \frac{x-1}{x} & (x > 1) \end{cases}$$

Potrebujeme ešte urobiť skúšku.

Skúška:

Ako?

(i) $y=0 \quad x>0$

$$f(0)=f(0)$$

(ii) $0 \leq y \leq 1 \quad f(y)=0$

$$f\left(\frac{xy}{x+y}\right)=0 \qquad \frac{xy}{x+y}<1 \text{ (prečo?)}$$

platí

(iii) $y>1$ + $\dfrac{xy}{x+y} \leq 1$

$$f\left(x\cdot\frac{(y-1)}{y}\right)\frac{(y-1)}{y}=f\left(\frac{xy}{x+y}\right)$$

$$f(xf(y))$$

$\overset{\shortparallel}{0}$ $\quad$ $\overset{\shortparallel}{0}$

$xy \leq x+y$

$$x \leq \frac{y}{y-1}=f(y)$$

$$x\frac{(y-1)}{y} \leq 1$$

$$xf(y) \leq 1$$

(iv) $y>1$ ale $\dfrac{xy}{x+y} \geq 1$

$$xf(y)>1$$

$$\frac{\frac{x(y-1)}{y}-1}{\frac{x(y-1)}{y}}\cdot\frac{y-1}{y} \overset{?}{=} \frac{\frac{xy}{x+y}-1}{\frac{xy}{x+y}}=\frac{xy-x-y}{xy}$$

$$\frac{xy-x-y}{xy-x}\cdot\frac{y-1}{y}=\frac{xy-x-y}{xy}$$

o.k.

PRÍKLAD:

$f: R^+ \to R^+$,

$$x^2(f(x)+f(y)) = (x+y)f(f(x)y).$$

Riešenie: $x = y$

$$2x^2 f(x) = 2x f(f(x)x) \quad x \neq 0$$

$$x f(x) = f(x f(x))$$

Ak $z = x f(x)$ $\quad \underline{z = f(z)}$? Nepomôže nám to!

Položme $x = y = 1$

$$\underline{f(1) = f(f(1))}$$

Položme $x = f(1)$, $y = 1$

$$f^2(1)\,(\underbrace{f(f(1))}_{=f(1)} + f(1)) = [f(1)+1]\,\underbrace{f(f(1))}_{f(1)}$$

$$2f^3(1) + \cancel{f^2(1)} = f^2(1) + f(1)$$

$$2x^3 - x^2 - x = 0$$

má tri reálne korene $x = 0$

$$2x^2 - x - 1 = 0 \quad x_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4} \begin{matrix} \nearrow 1 \\ \searrow -\frac{1}{2} \end{matrix}$$

$f(1) = x > 0 \Rightarrow$

$$\underline{\underline{f(1) = 1}}$$

Položíme

$x = 1 \qquad y$ ľub.

$$1 + f(y) = (1+y)f(y)$$

$$1 = y f(y)$$

$$f(y) = \frac{1}{y}$$

Skúška:

$$x^2 \left(\frac{1}{x} + \frac{1}{y}\right) = x+y \cdot \left(\frac{y}{x}\right)^{-1} = \frac{(x+y)x}{y}$$

$$\frac{x^2 \cdot (x+y)}{xy}$$

✓

PRÍKLAD: $f: R^+ \to R^+$

$$f(x f(y)) = f(xy) + x$$

Riešenie: $x = y = 1$

$$f(f(1)) = f(1) + 1$$

$x = y$

$$f(x f(x)) = f(x^2) + x$$

$x = 1 \quad y$ ľub.?

$$f(f(y)) = f(y) + 1$$

$$f^2(y) - f(y) - 1 = 0$$

$x = f(1) \quad , y = 1$?

$$f(f^2(1)) = f(f(1)) + f(1)$$

TOTO NEJDE! (PREČO?)

$$a^3 - a^2 - a = 0$$

$$a(a^2 - a - 1) = 0$$

$f(1) = 1$?

$a = 0 \qquad a = \frac{1}{\;} \quad -\frac{1}{2}$

INÁČ?

$x = f(x)$

$$f(f(x)f(y)) = f(f(x)y) + f(x)$$

$\Downarrow$

$$f(f(x)y) = f(f(x)f(y)) - f(x)$$

zamením x a y: $\quad //$

$$f(f(x)y) = f(yx) + y$$

$$f(f(x)f(y)) = f(x) + y + f(yx)$$

$\downarrow$

táto strana sa nezmení po výmene x a y

$$\Rightarrow \quad f(yx) + y + f(x) = f(yx) + x + f(y)$$

$\Downarrow$

$$f(x) - x = f(y) - y \qquad \forall x, y$$

$$\Rightarrow f(x) = x + c$$

po dosadení do rovnice dostanem, že $c = 1$!

PRIKLAD:

$f: R^+ \to R^+$,

$$f(x)\, f(y) = f(y) \cdot f(x \cdot f(y)) + \frac{1}{xy}$$

Riešenie:

$x = y = 1$

$$f^2(1) = f(1) \cdot f(f(1)) + 1$$

$$\boxed{f(1) = p}$$

$$p\, f(p) + 1 = f^2(p)$$

nie.

Iné $x = y$

$$f^2(x) = f(x) \cdot f(x f(x)) + \frac{1}{x^2}$$

Inač...

$$\boxed{f(x f(y)) = f(x) - \frac{1}{xy f(y)}}$$

$x = 1$ $\quad f(f(y)) = f(1) - \frac{1}{y f(y)} = p - \frac{1}{y f(y)}$

$y = 1$ $\quad f(xp) = f(x) - \frac{1}{xp}$

$x = 1$ $\quad \boxed{f(p) = f(1) - \frac{1}{p} = p - \frac{1}{p}}$

Ďalej: Zvolím $f(p) = p - \frac{1}{p}$

$x = p = f(1)$

$$f(p\,f(y)) = f(p) - \frac{1}{p y f(y)} = p - \frac{1}{p} - \frac{1}{p y f(y)}$$

$\parallel$

upravíme a dostaneme:

$$f(p \cdot f(y)) = f(f(y)) - \frac{1}{\cancel{y}p f(y)} = p - \frac{1}{y f(y)} - \frac{1}{p\cancel{y} f(y)}$$

Porovnáme a dostaneme

$$\cancel{p} + \frac{1}{p} + \frac{1}{p y f(y)} = \cancel{p} + \frac{1}{y f(y)} + \frac{1}{p f(y)} \quad \Big| \cdot y p \cdot f(y)$$

$$f(y) + \frac{1}{y} = \frac{p}{y} + 1$$

$$f(y) = 1 + \frac{p-1}{y}$$

dosadíme a zistíme, že

má platiť $(a-1)^2 = 1$

$\underline{a = 2}$.