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Recursion operators, nonlocal symmetries and related structures for some integrable systems

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Contents

1. Introduction

The present thesis is based on two independent papers [1] and [2] which constitute its body. The common subjects are nonlinear integrable evolution partial differential equations, their recursion operators, hierarchies of symmetries and conservation laws.

The geometrical theory of PDEs has a long history, see e.g. [KV] for more details. It had originated in the works by Lie [LI1, LI2, LI3], Bäcklund [BÄ], Monge [MO], Darboux [DA], Bianchi [BI] and later by Cartan [CA]. A major advance in this theory was made with the introduction of the notion of jet bundles by Ehresmann [EH]; the latter provides an adequate language for the theory in question.

In 1960s the first examples of integrable nonlinear partial differential systems were discovered [GG, MI, MG, SG, GA, KM, GK, BC, NP] and the associated (bi-)Hamiltonian structures were found [GA, ZF, MR]. It became clear that integrable systems possess infinite series of (possibly nonlocal) symmetries, and, moreover, existence of infinitely many such symmetries implies integrability [BL, O2, MS].

In 1977 Olver [O1] introduced the so-called recursion operators. They are linear operators (typically pseudodifferential ones in the case of nonlinear systems) on the algebra of differential functions which map the set of characteristics of symmetries into itself, see Sections 3 and 4 below and references therein for details. Thus, recursion operators allow us to generate infinite families of symmetries from a suitable seed symmetry, and hence such operators play an important role in establishing integrability. Below this is illustrated by the example of the Mikhailov–Novikov–Wang system studied in our paper [1], where the existence of a recursion operator enabled us to construct infinite hierarchies of commuting symmetries (and of cosymmetries as well) and therefore complete establishing integrability for the system under study.

The notion of (generalized) symmetry was further generalized to that of a nonlocal symmetry and this has led to the concept of a differential covering [VK, KR], which proved to play an important role in the geometry of PDEs. The recursion operators were subsequently interpreted in the terms of coverings by Marvan [MA] (see e.g. also [GU]) as the Bäcklund autotransformations of linearized diffieties.

In fact, integrable systems in addition to local symmetries usually possess infinite hierarchies of (shadows of) nonlocal symmetries. It can be even argued that precisely such *nonlocal* hierarchies are the most common feature of known today integrable partial differential systems in any number of independent variables, because (nonlinearizable non-overedetermined) integrable systems in more than two independent variables are generally believed to admit no infinite hierarchies of local symmetries, cf. e.g. [BL] and references therein.

However, surprisingly enough, to date it was not known whether the celebrated Krichever–Novikov equation [KN], which is well known to be integrable and possesses infinitely many local generalized symmetries and two recursion operators, see e.g. [DS] and references therein, has any nonlocal symmetries at all. In the second paper [2] of the present thesis we address this open problem by constructing new infinite hierarchies of shadows of nonlocal symmetries and cosymmetries for the Krichever–Novikov equation using the inverse $\mathcal{R}_1^{-1}$ of the fourth-order recursion operator of the latter. Moreover, we also tackle the problem, which was pointed out in [DS], of how to apply the composition $\mathcal{R}_2 \circ \mathcal{R}_1^{-1}$, where $\mathcal{R}_2$ is the sixth-order recursion operator for the Krichever–Novikov equation, to the known symmetries of the equation in question.

The presentation in Sections 2–4 closely follows [BV, KV, MA].

2. Jet spaces

In this section we introduce the definition of jet space and review some important geometric structures related to it. In what follows we tacitly assume that all objects are smooth unless otherwise explicitly stated.

Consider an m-dimensional locally trivial bundle $\pi : E \to M$ over an n-dimensional manifold M and denote by $\Gamma(\pi)$ the set of all sections $s : M \to E$. The set $\Gamma(\pi)$ forms a module over the algebra $C^\infty(M)$ of smooth functions on M. Recall that by definition $\pi \circ s = \mathrm{id}_M$, i.e. the section s takes a point $x \in M$ to some point $s(x) \in E_x$, where the set $E_x = \pi^{-1}(x) \subset E$ is the so-called *fiber* of E over x. Note that in what follows all bundles under consideration are tacitly assumed to be vector bundles. This means that for each $x \in M$ the fiber E_x is endowed with the structure of an m-dimensional vector space and the gluing functions are linear transformations.

Two sections $s_1, s_2 \in \Gamma(\pi)$ are said to be *k-equivalent* at a point $x \in M$ if their graphs are tangent to each other with order k at the point $s_1(x) = s_2(x) \in E$. The set of equivalence classes of sections will be denoted by J_x^k and called the *space of k-jets* of the bundle π at the point x. A point of this space will be denoted by $[s]_x^k$. The set

$$J^k(\pi) = \bigcup_{x \in M} J_x^k = \left\{ [s]_x^k \mid x \in M, s \in \Gamma(\pi) \right\}$$

is called the *space of k-jets of the bundle π*. Moreover, we define the projections

$$\pi_k : J^k(\pi) \to M, \qquad [s]_x^k \mapsto x$$

and

$$\pi_{k,l} : J^k(\pi) \to J^l(\pi), \qquad [s]_x^k \mapsto [s]_x^l, \quad k \geq l.$$

Then, for any section $s \in \Gamma(\pi)$ the map

$$j_k(s) : M \to J^k(\pi), \qquad x \mapsto [s]_x^k$$

is a smooth section of π_k which is called the *k-jet* of s.

Let $\mathcal{U} \subset M$ be a coordinate neighborhood such that the bundle π becomes trivial over $\mathcal{U}$. Let $x^1, \dots, x^n, u^1, \dots, u^m$ be an adapted coordinate system in the bundle π over a neighborhood $\mathcal{U}$ of the point $x \in M$. Consider the set $\pi_k^{-1}(\mathcal{U}) \subset J^k(\pi)$. Then the functions $u_I^j : J^k(\pi) \to \mathbb{R}$, I being a multi-index, defined by the formula

$$u_I^j([s]_x^k) = \frac{\partial^{|I|} s^j}{\partial (x^1)^{i_1} \ldots \partial (x^n)^{i_n}}, \qquad j = 1, \ldots, m, \quad |I| \le k$$

complete local coordinates $x^1, \ldots, x^n, u^1, \ldots, u^m$ to the local coordinates on the set $\pi_k^{-1}(\mathcal{U})$. We call the coordinates u_I^j *canonical* (or *special*) *coordinates* associated to the adapted coordinate system (x^i, u^j). Thus, the set $J^k(\pi)$ is endowed with a structure of a smooth manifold.

For a given bundle π one can consider all jet manifolds $J^k(\pi)$, $k = 0, 1, \ldots$, arranging them one over another as a tower, see [BV].

Jets and projections	**Coordinates in jets**
$\ldots$	$\ldots$
$\downarrow \pi_{k+2,k+1}$	
$J^{k+1}(\pi)$	x^i, u^j, u_I^j, $\lvert I\rvert \le k+1$
$\downarrow \pi_{k+1,k}$	
$J^k(\pi)$	x^i, u^j, u_I^j, $\lvert I\rvert \le k$
$\downarrow \pi_{k,k-1}$	
$\ldots$	$\ldots$
$\downarrow \pi_{1,0}$	
$J^0(\pi) = E$	x^i, u^j
$\downarrow \pi$	
M	x^i

For any point $x \in M$ choose a sequence of points $\theta_l \in J^l(\pi)$, $l = 0, 1, \ldots$, such that $\pi_{l+1,l}(\theta_{l+1}) = \theta_l$ and $\pi(\theta_0) = x$. Then one can choose a local section s of the bundle π such that $\theta_l = [s]_x^l$ for any l. Thus the whole sequence $\{\theta_l\}$ contains information on all partial derivatives of the section s at x. Denote by $J^\infty(\pi)$ the set of all such sequences. The points of the space $J^\infty(\pi)$ can be thought of as the equivalence classes of sections of the bundle π tangent to each other with infinite order.

For any point $\theta_\infty = \{x, \theta_k\}_{k \in \mathbb{N}} \in J^\infty(\pi)$, we define $\pi_{\infty,k}(\theta_\infty) = \theta_k$ and $\pi_\infty(\theta_\infty) = x$. Then for all $k \ge l \ge 0$ we have the equalities $\pi_k \circ \pi_{\infty,k} = \pi_\infty$ and $\pi_{k,l} \circ \pi_{\infty,k} = \pi_{\infty,l}$. Moreover, if s is a section of the bundle π then the mapping $j_\infty(s) : M \to J^\infty(\pi)$ is defined by the equality $j_\infty(s)(x) = \{x, [s]_x^k\}_{k \in \mathbb{N}}$. Then one has the following identities: $\pi_{\infty,k} \circ j_\infty(s) = j_k(s)$

and $\pi_\infty \circ j_\infty = \mathrm{id}_M$. The section $j_\infty(s)$ of the bundle π_∞ is called an *infinite jet of the section $s \in \Gamma(\pi)$*.

Local coordinates arising in $J^\infty(\pi)$ over a neighborhood $\mathcal{U} \subset M$ are $x^1, \ldots, x^n$ together with all functions u^j_I, where the multi-index I is such that $|I|$ is of an arbitrary non-negative finite value; here and below $u^j_0 = u^j$. Thus the set $J^\infty(\pi)$ is endowed with a structure of an infinite-dimensional smooth manifold.

The bundle $\pi_\infty : J^\infty(\pi) \to M$ is called the *bundle of infinite jets*, while the space $J^\infty(\pi)$ is called the *manifold of infinite jets* of the bundle π.

Smooth functions on $J^\infty(\pi)$ are defined as elements of the filtered algebra $\mathcal{F}(\pi) = \bigcup_k \mathcal{F}_k(\pi)$, where $\mathcal{F}_k(\pi) = C^\infty(J^k(\pi))$. Thus, in the canonical coordinates, they are of the form $f = f(x, u^{(k)})$, where $x = (x^1, \ldots, x^n)$ and $u^{(k)} = \{u^j_I \,|\, |I| \leq k, j = 1, \ldots, m\}$ is the finite set of the dependent variables and their derivatives up to some finite order k. We will call such functions f *differential functions*.

A tangent vector X_θ to the manifold $J^\infty(\pi)$ at the point θ is defined as the set $\{X_x, X_{\theta_k}\}_{k\in\mathbb{N}}$ of the tangent vectors to the manifolds M and $J^k(\pi)$ at the points $x = \pi_\infty(\theta)$ and $\theta_k = \pi_{\infty,k}(\theta)$, such that $(\pi_k)_*(X_{\theta_k}) = X_x$ and $(\pi_{k+1,k})_*(X_{\theta_{k+1}}) = X_{\theta_k}$. In the canonical coordinates on $\pi_\infty^{-1}(\mathcal{U}) \subset J^\infty(\pi)$, any tangent vector X_θ is represented as an infinite sum

$$X_\theta = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} + \sum_{|I|\geq 0} \sum_{j=1}^m a^j_I \frac{\partial}{\partial u^j_I}, \tag{1}$$

where $a^i, a^j_I \in \mathbb{R}$. We also have projections

$$(\pi_{\infty,k})_*(X_\theta) = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^k \sum_{j=1}^m a^j_I \frac{\partial}{\partial u^j_I}, \qquad (\pi_\infty)_*(X_\theta) = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}.$$

A *vector field* on $J^\infty(\pi)$ is defined as a (smooth) assignment $X : \theta \mapsto X_\theta$, where $\theta \in J^\infty(\pi)$. We will denote the set of all vector fields on $J^\infty(\pi)$ by $\mathcal{X}(\pi)$. Just as in the case of finite-dimensional manifolds, a vector field $X \in \mathcal{X}(\pi)$ can be considered as a derivation of the algebra $\mathcal{F}(\pi)$, i.e. an $\mathbb{R}$-linear mapping $X : \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ such that

$$X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$$

for all $f_1, f_2 \in \mathcal{F}(\pi)$. In the canonical coordinates, a vector field $X \in \mathcal{X}(\pi)$ is represented as an infinite sum

$$X = \sum_i a^i \frac{\partial}{\partial x^i} + \sum_{I,j} a^j_I \frac{\partial}{\partial u^j_I}, \qquad a^i, a^j_I \in \mathcal{F}(\pi).$$

Definition 2.1. [BV] A vector field $X \in \mathcal{X}(\pi)$ is called *vertical*, if

$$X(\pi_\infty^*(f)) = 0$$

for any function $f \in C^\infty(M)$.

We denote the set of all vertical vector fields on $J^\infty(\pi)$ by $\mathcal{X}^v(\pi)$. In the canonical coordinates, the vertical vector fields are characterized by the fact that all their coefficients at $\partial/\partial x^i$ vanish.

Denote by $\Lambda^i(\pi_k) = \Lambda^i(J^k(\pi))$ the module of i-forms on $J^k(\pi)$. We define the module $\Lambda^i(\pi)$ of i-forms on $J^\infty(\pi)$ by the formula $\Lambda^i(\pi) = \bigcup_k \Lambda^i(\pi_k)$. In the canonical coordinates, any form $\omega \in \Lambda^i(\pi)$ can be expressed as

$$\omega = \sum_{\alpha+\beta=i} \varphi^{I_1,\ldots,I_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta} dx^{i_1} \wedge \ldots \wedge dx^{i_\alpha} \wedge du^{j_1}_{I_1} \wedge \ldots \wedge du^{j_\beta}_{I_\beta},$$

where $|I_1|, \ldots, |I_\beta| \le k$ and $\varphi^{I_1,\ldots,I_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta}$ are differential functions. Note that if we set $\Lambda^*(\pi) = \bigoplus_{i=0}^\infty \Lambda^i(\pi)$, then the wedge product $\wedge$ and the exterior derivative d are defined in the usual manner on $\Lambda^*(\pi)$ and enjoy their usual properties.

Let $X \in \mathcal{X}(\pi)$ and $\omega \in \Lambda^i(\pi)$. We define the *inner product* $i_X : \Lambda^i(\pi) \to \Lambda^{i-1}(\pi)$ by

$$(i_X\omega)_\theta = i_{X_{\theta_k}} \omega_{\theta_k}. \tag{2}$$

By the definition of $\Lambda^i(\pi)$, there always exists a number k such that $\omega \in \Lambda^i(\pi_k)$, and we have $(\pi_{k',k})_*(X_{\theta_{k'}}) = X_{\theta_k}$ for any $k' \ge k$. Hence, the equality $i_{X_{\theta_{k'}}}(\pi^*_{k',k}\omega)_{\theta_{k'}} = i_{X_{\theta_k}}\omega_{\theta_k}$ holds, and thus the operation i_X is well-defined by (2).

Definition 2.2. [BV] A form $\omega \in \Lambda^*(\pi)$ is called *horizontal* if $i_X\omega = 0$ for any vertical vector field $X \in \mathcal{X}^v(\pi)$.

We denote by $\Lambda^i_0(\pi)$ the space of all horizontal i-forms on $J^\infty(\pi)$. Any horizontal form ω is representable in the canonical coordinates as

$$\omega = \sum \varphi_{i_1,\ldots,i_\alpha} dx^{i_1} \wedge \ldots \wedge dx^{i_\alpha},$$

where $\varphi_{i_1,\ldots,i_\alpha} \in \mathcal{F}(\pi)$.

Let $\theta \in J^\infty(\pi)$. Then the graphs of all sections $j_\infty(s), s \in \Gamma(\pi)$, that pass through the point θ have a common n-dimensional tangent plane $\mathcal{C}_\theta$, the so-called *Cartan plane*. The corespondence $\mathcal{C} : \theta \mapsto \mathcal{C}_\theta$ is an integrable n-dimensional distribution on $J^\infty(\pi)$ which is called the *Cartan distribution.*

For any vector field X on M there is a unique field $\mathcal{C}X$ on $J^\infty(\pi)$ such that for any section $s : M \to E$ of π and any $f \in \mathcal{F}(\pi)$ we have $X(j^*_\infty(s)f) = j^*_\infty(s)(\mathcal{C}X(f))$. In such a way one obtains a connection in the bundle π_∞ called the *Cartan connection.* This connection is flat, i.e.,

$$\mathcal{C}[X,Y] = [\mathcal{C}X, \mathcal{C}Y] \tag{3}$$

for all vector fields X, Y on M. By virtue of (3), the space $\mathcal{C}\mathcal{X}(\pi)$ of all vector fields lying in the Cartan distribution is a Lie subalgebra in $\mathcal{X}(\pi)$. Vector fields belonging to $\mathcal{C}\mathcal{X}(\pi)$ are called the *Cartan fields.*

In the canonical coordinate system (x^i, u^j_I) on $J^\infty(\pi)$, we have

$$\mathcal{C} : \frac{\partial}{\partial x^i} \mapsto D_i = \frac{\partial}{\partial x^i} + \sum_{I,j} u^j_{Ii} \frac{\partial}{\partial u^j_I}.$$

The fields D_i are called *total derivatives* and they span the Cartan distribution.

3. Differential equations

Let Δ be a section of a finite-dimensional vector bundle ξ over $J^\infty(\pi)$. Then $\Delta = 0$ is nothing but a system $\mathcal{E}$ of partial differential equations written a coordinate-free way, see e.g. [KV].

In local coordinates such a system (in n independent variables x^i and m dependent variables u^j) has the form

$$\Delta_s(x, u^{(k)}) = 0, \qquad s = 1, \ldots, l, \tag{4}$$

Then (4) determines a submanifold

$$\mathcal{E} = \{\theta_k \in J^k(\pi) \mid \Delta_1(\theta_k) = \ldots = \Delta_l(\theta_k) = 0\} \tag{5}$$

in the jet space $J^k(\pi)$ of a vector bundle $\pi : E \to M$, such that $\dim E = m+n$ and $\dim M = n$. For the sake of simplicity we shall call a submanifold $\mathcal{E}$ an *equation* even though it can actually be a system of several PDEs.

Without loss of generality we assume that the projection $\pi_{\infty,0} : \mathcal{E}^\infty \to J^0(\pi)$ is surjective, i.e. that (4) does not contain equations of zero order. Obviously, the Cartan plane $\mathcal{C}_\theta \subset T_\theta(\mathcal{E}^\infty)$ at every point $\theta \in \mathcal{E}^\infty$, so the dimension of the Cartan distribution on a diffiety is equal to n.

We can extend (4) to a larger system

$$D_J(\Delta_s(x, u^{(k)})) = 0 \tag{6}$$

for all multi-indices $J = (j_1, \ldots, j_n)$ and $s = 1, \ldots, l$, where $D_J = D_{x^1}^{j_1} \circ \ldots \circ D_{x^n}^{j_n}$. Thus (6) includes all differential consequences of (4). Consider the submanifold $\mathcal{E}^\infty$ defined by (6) in the jet space $J^\infty(\pi)$. Then the (smooth) solutions of (4) are the sections of π whose infinite jets lie in $\mathcal{E}^\infty$. In other words, the solutions of (4) are the maximal integral submanifolds of the Cartan distribution restricted to $\mathcal{E}^\infty$. We call such a submanifold $\mathcal{E}^\infty$ endowed with the Cartan distribution a *diffiety*. A diffiety is generally of infinite dimension. For a coordinate-free definition of $\mathcal{E}^\infty$ see [BV].

Denote by $\mathcal{C}\Lambda^1(\mathcal{E}^\infty)$ the set of all Cartan forms on $\mathcal{E}^\infty$, i.e. the set of all one-forms on $\mathcal{E}^\infty$ which are annihilated by vectors of the Cartan distribution $\mathcal{C}(\mathcal{E})$ at every point $\theta \in \mathcal{E}^\infty$. We define the Lie $\mathbb{R}$-algebra

$$\operatorname{sym} \mathcal{E} = \mathcal{X}_\mathcal{C}(\mathcal{E}^\infty)/\mathcal{C}\mathcal{X}(\mathcal{E}^\infty),$$

where

$$\mathcal{C}\mathcal{X}(\mathcal{E}^\infty) = \{X \in \mathcal{X}(\mathcal{E}^\infty) \,|\, i_X\omega = 0, \forall \omega \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty)\},$$

and

$$\mathcal{X}_\mathcal{C}(\mathcal{E}^\infty) = \{X \in \mathcal{X}(\mathcal{E}^\infty) \,|\, [X, \mathcal{C}\mathcal{X}(\mathcal{E}^\infty)] \subset \mathcal{C}\mathcal{X}(\mathcal{E}^\infty)\}.$$

Definition 3.1. [BV] We call elements of $\operatorname{sym} \mathcal{E}$ *higher*[1] symmetries of the equation $\mathcal{E}$.

[1] Such symmetries are also known as *generalized*, see e.g. [O2].

To describe the Lie algebra $\operatorname{sym}\mathcal{E}$ in a more explicit fashion, denote by $\mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty)$ the set of all vector fields $X \in \mathcal{X}_{\mathcal{C}}(\mathcal{E}^\infty)$ such that $X(\pi^*_\infty(f)) = 0$ for any $f \in C^\infty(M)$. For any $X \in \mathcal{X}_{\mathcal{C}}(\mathcal{E}^\infty)$ it is possible to find the vector field $\mathcal{C}X \in \mathcal{C}\mathcal{X}(\mathcal{E}^\infty)$ such that $X = X^v + \mathcal{C}X$, where $X^v \in \mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty)$, see [BV] for more details. Thus we have the correspondence $X \mapsto X^v$, which determines the mapping

$$v : \mathcal{X}_{\mathcal{C}}(\mathcal{E}^\infty) \to \mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty). \tag{7}$$

Lemma 3.1. [BV] *The mapping v is a projector, i.e., $X^v = X$ for every $X \in \mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty)$. Moreover,* $\ker v = \mathcal{C}\mathcal{X}(\mathcal{E}^\infty)$.

It follows from Lemma 3.1 that

$$\mathcal{X}_{\mathcal{C}}(\mathcal{E}^\infty) = \mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty) \oplus \ker v,$$

and the latter equality induces an isomorphism

$$\operatorname{sym}\mathcal{E} \simeq \mathcal{X}^v_{\mathcal{C}}(\mathcal{E}^\infty).$$

It is easily seen that the quantity $(\pi_{\infty,0})_*X$ can be identified with an element $F_X \in \Gamma(\pi^*(\pi))$, where $\pi^*(\pi)$ is the pullback of π to $J^\infty(\pi)$. The correspondence $X \mapsto F_X$ is bijective (see e.g. [KV]).

Definition 3.2. [KV, BV] The section $F_X \in \Gamma(\pi^*(\pi))$ is called a *characteristic* of the symmetry X, while the symmetry corresponding to a section $F \in \Gamma(\pi^*(\pi))$ is called an *evolutionary derivation* associated to F and is denoted by $\mathbf{E}_F$. The quantity $\mathbf{v}_F = (\pi_{\infty,0})_*\mathbf{E}_F$ is called an *evolutionary vector field*.

In the canonical coordinates a section $F \in \Gamma(\pi^*(\pi))$ becomes a vector-valued differential function $F = (F^1(x, u^{(k)}), \dots, F^m(x, u^{(k)}))$. Then the corresponding evolutionary derivation is

$$\mathbf{E}_F = \sum_{|I|\geq 0} \sum_{j=1}^m D_I(F^j) \frac{\partial}{\partial u^j_I}, \tag{8}$$

and it readily follows from (8) that

$$\mathbf{v}_F = \sum_{j=1}^m F^j \frac{\partial}{\partial u^j}, \tag{9}$$

It is readily seen that for any evolutionary derivations $\mathbf{E}_F$ and $\mathbf{E}_G$ there exists an m-component vector function $\{F, G\}$ such that

$$\mathbf{E}_{\{F,G\}} = [\mathbf{E}_F, \mathbf{E}_G].$$

This function is called [BV] the *Jacobi bracket* of F and G.

Put $\Delta = (\Delta_1(x, u^{(k)}), \dots, \Delta_l(x, u^{(k)}))$. The *Fréchet* (or *directional*) *derivative* or *linearization* of Δ is (see e.g. [BL, KV, O2, BV]) the $l \times m$ matrix differential operator ℓ_Δ with entries

$$(\ell_\Delta)_{ij} = \sum_{I,j} \frac{\partial \Delta_i}{\partial u^j_I} D_I.$$

Theorem 3.2. [BV] *A vector-valued function $U : J^\infty(\pi) \to \mathbb{R}^m$ is the characteristic of a higher symmetry of an equation $\mathcal{E}$, i.e., of (4), if and only if*

$$\mathbf{E}_U(\Delta_s) = 0 \tag{10}$$

for every $\theta_\infty \in \mathcal{E}^\infty$ and $s = 1, \dots, l$, or equivalently

$$\ell_\Delta|_{\mathcal{E}^\infty}(U) = 0. \tag{11}$$

Moreover, the Lie algebra $\operatorname{sym}\mathcal{E}$ *is isomorphic to the space of solutions of the system (11) endowed with the Jacobi bracket.*

The equations (11) are called the *determining equations* for higher symmetries of $\mathcal{E}$; they express vanishing of the linearization of the system (4), restricted to $\mathcal{E}^\infty$, along $U = (U^1, \dots, U^m)$. For coordinate-free versions of the above theorem and of the definition of linearization cf. e.g. [BV, KV].

4. Coverings, nonlocal symmetries and recursion operators

Definition 4.1. [BV] We shall say that a *covering* $\tau : \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ of the equation $\mathcal{E}$ is given, if the following objects are fixed:

1. A smooth manifold $\widetilde{\mathcal{E}}$, infinite-dimensional in general.
2. An n-dimensional integrable distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$.
3. A regular mapping τ of the manifold $\widetilde{\mathcal{E}}$ onto $\mathcal{E}^\infty$ such that for any point $\theta \in \widetilde{\mathcal{E}}$ the tangent mapping $\tau_{*,\theta}$ is an isomorphism of the plane $\widetilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ of the equation $\mathcal{E}^\infty$ at the point $\tau(\theta)$.

The dimension of the bundle τ is called the dimension of the corresponding covering. It follows from the definition that the mapping τ takes any n-dimensional integral manifold $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{E}}$ of the distribution $\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{C}}_\theta\}_{\theta \in \widetilde{\mathcal{E}}}$ to an n-dimensional integral manifold $\mathcal{U} = \tau(\widetilde{\mathcal{U}}) \subset \mathcal{E}^\infty$ of the Cartan distribution on $\mathcal{E}^\infty$, i.e., to a solution of the equation $\mathcal{E}$. Conversely, if $\mathcal{U} \subset \mathcal{E}^\infty$ is a solution of the equation $\mathcal{E}$, then the restriction of the distribution $\widetilde{\mathcal{C}}$ to the inverse image $\widetilde{\mathcal{U}} = \tau^{-1}(\mathcal{U}) \subset \widetilde{\mathcal{E}}$ is an integrable n-dimensional distribution.

The manifold $\widetilde{\mathcal{E}}$ and the mapping $\tau : \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ can be locally realized as the direct product $\mathcal{E}^\infty \times W$, where $W \subseteq \mathbb{R}^N$ is an open set, $0 < N \leq \infty$, and as the natural projection $\mathcal{E}^\infty \times W \to \mathcal{E}^\infty$ respectively. Then the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times W$ can be described by the system of vector fields

$$\widetilde{D}_i = D_i + \sum_{j=1}^{N} X_{ij} \frac{\partial}{\partial w_j}, \quad i = 1, \dots, n, \tag{12}$$

where $X_i = X_{ij}\partial/\partial w_j$, $X_{ij} \in C^\infty(\widetilde{\mathcal{E}})$, are τ-vertical fields on $\widetilde{\mathcal{E}}$, and $w_1, w_2, \dots$ are standard coordinates on $\mathbb{R}^N$.

Locally $\widetilde{\mathcal{E}}$ is nothing but (6) combined with the following equations:

$$\frac{\partial w_i}{\partial x^j} = X_{ji}, \quad i = 1, \dots, N, \quad j = 1, \dots, n. \tag{13}$$

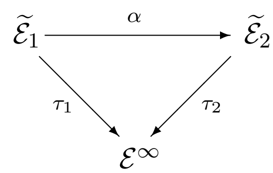
The conditions $[\widetilde{D}_i, \widetilde{D}_j] = 0$ are equivalent to the equations

$$\widetilde{D}_i(X_{jk}) = \widetilde{D}_j(X_{ik}) \tag{14}$$

for all $i, j = 1, \dots, n$, $1 \le k \le N$, which must hold on $\widetilde{\mathcal{E}}^\infty$.

Relations (14) constitute a system of differential equations in functions X_{ij} describing all possible N-dimensional coverings over the equation $\mathcal{E}$. The coordinates w_i are called *nonlocal variables.*

Definition 4.2. [BV] Two coverings $\tau_i : \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i = 1, 2$, are called *equivalent* if there exists a diffeomorphism $\alpha : \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_2$ such that the diagram



is commutative and $\alpha_*(\widetilde{\mathcal{C}}^1_y) = \widetilde{\mathcal{C}}^2_{\alpha(y)}$ for all points $y \in \widetilde{\mathcal{E}}_1$.

Let $\tau : \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering over the equation $\mathcal{E}$. A *nonlocal symmetry* of the equation $\mathcal{E}$ is by definition a local symmetry of the object $\widetilde{\mathcal{E}}$. Nonlocal symmetries in the covering $\tau : \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ will be called *symmetries of type* τ, or *nonlocal τ-symmetries.*

Definition 4.3. [BV] The Lie algebra of *nonlocal τ-symmetries* of the equation $\mathcal{E}$ is the quotient Lie algebra

$$\mathrm{sym}_\tau \mathcal{E} = \mathcal{X}_{\mathcal{C}}(\widetilde{\mathcal{E}})/\mathcal{C}\mathcal{X}(\widetilde{\mathcal{E}}),$$

where

$$\mathcal{C}\mathcal{X}(\widetilde{\mathcal{E}}) = \left\{ \sum_{i=1}^n \varphi_i \widetilde{D}_i \mid \varphi_i \in C^\infty(\widetilde{\mathcal{E}}) \right\},$$

while $\mathcal{X}_{\mathcal{C}}(\widetilde{\mathcal{E}})$ consists of vector fields X on $\widetilde{\mathcal{E}}$ such that $[X, \mathcal{C}\mathcal{X}(\widetilde{\mathcal{E}})] \subset \mathcal{C}\mathcal{X}(\widetilde{\mathcal{E}})$.

If coverings τ_1 and τ_2 are equivalent, then the Lie algebras of nonlocal symmetries $\mathrm{sym}_{\tau_1}\mathcal{E}$ and $\mathrm{sym}_{\tau_2}\mathcal{E}$ are isomorphic [BV].

Consider a vector-valued differential function $U : \widetilde{\mathcal{E}} \to \mathbb{R}^m$ and define the evolutionary derivation associated to U on $\widetilde{\mathcal{E}}$ by the formula

$$\widetilde{\mathbf{E}}_U = \sum_{|I|\ge 0} \sum_{j=1}^m \widetilde{D}_I(U^j) \frac{\partial}{\partial u^j_I}. \tag{15}$$

Then $\widetilde{\mathbf{E}}_U$ is called a *τ-shadow of nonlocal symmetry* of $\mathcal{E}$ if the equality $\widetilde{\mathbf{E}}_U(\Delta_s) = 0$ holds for all $s = 1, \dots, l$ by virtue of (4), (13) and differential consequences thereof, see e.g. [KV]. Equivalently, the characteristic U must satisfy the equation $\widetilde{\ell}_\Delta|_{\mathcal{E}^\infty}(U) = 0$, where $\widetilde{\ell}_\Delta$ is the linearization of Δ naturally lifted to $\widetilde{\mathcal{E}}$.

In a similar fashion, the solutions of the adjoint system $\widetilde{\ell}^*_\Delta|_{\mathcal{E}^\infty}(\gamma) = 0$ will be called *τ-shadows of nonlocal cosymmetries* of $\mathcal{E}$, cf. e.g. [KV].

Theorem 4.1. [BV] *Let $\tau : \widetilde{\mathcal{E}}^\infty \times \mathbb{R}^N \to \widetilde{\mathcal{E}}^\infty$ be a covering of the equation $\mathcal{E}$. Then any nonlocal τ-symmetry of $\mathcal{E}$ is of the form*

$$\widetilde{\mathbf{E}}_{U,A} = \widetilde{\mathbf{E}}_U + \sum_{j=1}^{N} a_j \frac{\partial}{\partial w_j}. \tag{16}$$

Here $\widetilde{\mathbf{E}}_U$ is a τ-shadow of nonlocal symmetry and $A = (a_1, \dots, a_N)$ is a vector-valued differential function on $\widetilde{\mathcal{E}}$ satisfying the equations

$$\widetilde{D}_i(a_j) = \widetilde{\mathbf{E}}_{U,A}(X_{ij}), \quad i, j = 1 \dots, N, \tag{17}$$

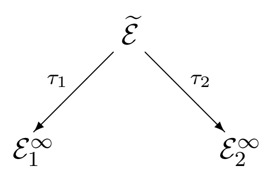
where X_{ij} are given in (12).

The coordinate-free versions of the above result and of the definition of nonlocal symmetries and shadows can be found in [BV].

Note that the system (17) may have no solution for a given U. Thus, not every τ-shadow of the equation $\mathcal{E}$ can be extended to a nonlocal τ-symmetry in the sense of Definition 4.3. However, for any given shadow one can construct a larger covering $\tilde{\tau}$ where it could be lifted to a full-fledged nonlocal symmetry, see e.g. §5.7 of [BV] and references therein for details.

A Bäcklund transformation between equations $\mathcal{E}_1$ and $\mathcal{E}_2$ is a system of differential relations in unknown functions u_1 and u_2 possessing the following property: if a function u_1 is a solution of the equation $\mathcal{E}_1$ and u_1 and u_2 satisfy the relations at hand, then u_2 is a solution of $\mathcal{E}_2$. Using the language of coverings, this definition reads as follows.

Definition 4.4. [BV] A *Bäcklund transformation* between equations $\mathcal{E}_1$ and $\mathcal{E}_2$ is the diagram



in which the mappings τ_1 and τ_2 are coverings.

If $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty$, then the Bäcklund transformation of the equation $\mathcal{E}$ is called a *Bäcklund autotransformation*. In the case of $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty = \mathcal{V}\mathcal{E}^\infty$, where $\mathcal{V}\mathcal{E}^\infty$ is the linearized version of $\mathcal{E}^\infty$ (see e.g. [MA] for details; there the object in question is denoted by VE), we arrive at the following definition of the recursion operator for $\mathcal{E}$.

Definition 4.5. [MA] A pair of coverings $\widetilde{\tau}_1, \widetilde{\tau}_2 : R \to \mathcal{V}\mathcal{E}^\infty$ is a *recursion operator* for $\mathcal{E}$ if the diagram

$$
\begin{array}{ccc}
 & R & \\
{\scriptstyle \widetilde{\tau}_1} \swarrow & & \searrow {\scriptstyle \widetilde{\tau}_2} \\
\mathcal{V}\mathcal{E}^\infty & & \mathcal{V}\mathcal{E}^\infty \\
{\scriptstyle \tau} \searrow & & \swarrow {\scriptstyle \tau} \\
 & \mathcal{E}^\infty &
\end{array}
$$

is commutative.

From now on we shall assume the bundle π to be trivial and work solely in the canonical local coordinates.

For instance, in the case of a (1+1)-dimensional system of evolution equations

$$\frac{\partial \vec{u}}{\partial t} = \vec{F}(x, t, \vec{u}, \vec{u}_x, \ldots, \vec{u}_{nx}) \tag{18}$$

in two independent variables x, t and l dependent variables u^i, where $\vec{u} = (u^1, \ldots, u^l)^T$, $\vec{F} = (F^1, \ldots, F^l)^T$, $\vec{u}_{jx} = \partial^j \vec{u}/\partial x^j$, $\vec{u}_{0x} \equiv \vec{u}$ the diagram from Definition 4.5 often has the form (cf. e.g. [S1]):

$$
\begin{array}{ccc}
 & \boxed{\begin{array}{c}
(W_j)_t = \displaystyle\sum_{k=1}^{n} \sum_{m=0}^{k-1} (-D_x)^m \left(\left(\frac{\partial \vec{F}}{\partial \vec{u}_{kx}} \right)^T \vec{\gamma}_j \right) \cdot D_x^{k-m-1}(\vec{U}) \\
(W_j)_x = \vec{\gamma}_j \cdot \vec{U} \qquad (j = 1, 2, \ldots) \\
\vec{U}_t = \ell_{\vec{F}}(\vec{U}) \\
\vec{u}_t = \vec{F}
\end{array}} & \\
{\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \widetilde{\tau}_2 : \vec{V} = \sum_{i=0}^{k} a_i D_x^i(\vec{U}) + \sum_j \vec{G}_j W_j} \\
\boxed{\begin{array}{c} \vec{U}_t = \ell_{\vec{F}}(\vec{U}) \\ \vec{u}_t = \vec{F} \end{array}} & & \boxed{\begin{array}{c} \vec{V}_t = \ell_{\vec{F}}(\vec{V}) \\ \vec{u}_t = \vec{F} \end{array}} \\
{\scriptstyle \pi} \searrow & & \swarrow {\scriptstyle \pi} \\
 & \boxed{\vec{u}_t = \vec{F}} &
\end{array}
\tag{19}
$$

where a_i are $l \times l$ matrix-valued functions, $\cdot$ is the standard scalar product in $\mathbb{R}^l$, and the superscript T indicates the transposed matrix.

Here $\vec{G}_j$ and $\vec{\gamma}_j$ are some fixed symmetries and cosymmetries and π is the canonical projection. Recall (see e.g. [BL]) that cosymmetries are solutions of the determining equation which is formally adjoint to the one for symmetries.

For (18) almost all their recursion operators known from the literature have the form (19); the latter however is usually rewritten as

$$\mathcal{R} = \sum_{i=0}^{p} a_i D_x^i + \sum_{j=1}^{q} \vec{G}_j \otimes D_x^{-1} \circ \vec{\gamma}_j, \tag{20}$$

Nevertheless, this approach is, in fact, somewhat inaccurate because it does not specify the derivatives of nonlocal variables W_j with respect to t, which may lead to incorrect results, cf. e.g. [GU]. Thus it is indeed helpful to rewrite the recursion operator $\mathcal{R}$ as a Bäcklund autotransformation for $\mathcal{VE}^\infty$ in the sense of Definition 4.5, cf. e.g. [GU, MA].

If a_i, G_j and γ_j in (20) are all local quantities (i.e., they are smooth functions of x, t, u and of finitely many derivatives of u with respect to x, see e.g. [MS]) then we call [MN] recursion operators of the form (20) *weakly nonlocal.*

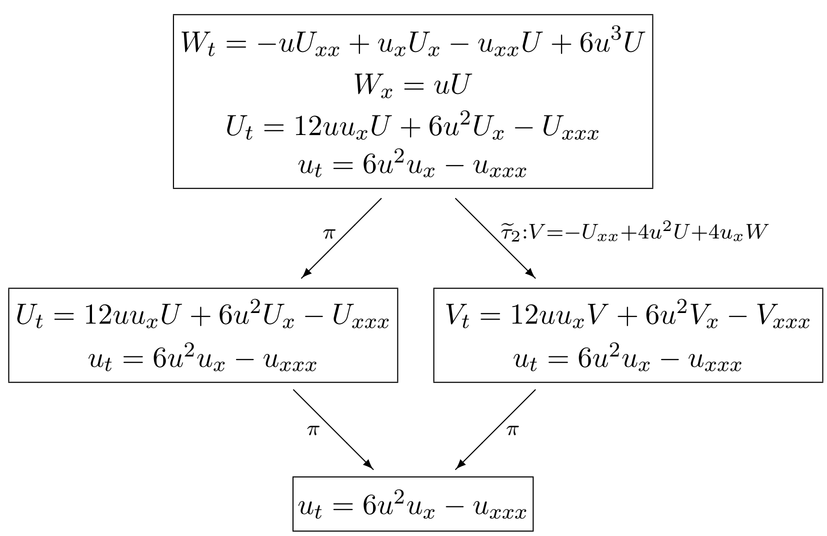
Example 1. Consider the modified Korteweg–de Vries (mKdV) equation

$$u_t = 6u^2 u_x - u_{xxx}. \tag{21}$$

It is well known that (21) admits a weakly nonlocal recursion operator (see e.g. [GU])

$$\mathcal{R} = -D_x^2 + 4u^2 + 4u_x D_x^{-1} \cdot u.$$

The commutative diagram (19) then has the following form:



5. On complete integrability of the Mikhailov–Novikov–Wang system

In the paper [1] we consider a new integrable two-component fifth-order system of evolution equations in two independent and two dependent variables recently found by Mikhailov *et al.* [MC] (see also [MS]):

$$
\begin{aligned}
u_t &= -\tfrac{5}{3}u_5 - 10vv_3 - 15v_1v_2 + 10uu_3 + 25u_1u_2 \\
&\quad -6v^2v_1 + 6v^2u_1 + 12uvv_1 - 12u^2u_1, \\
v_t &= 15v_5 + 30v_1v_2 - 30v_3u - 45v_2u_1 - 35v_1u_2 \\
&\quad -10vu_3 - 6v^2v_1 + 6v^2u_1 + 12u^2v_1 + 12vuu_1.
\end{aligned}
\tag{22}
$$

Here $u_i = \partial^i u/\partial x^i$, $v_j = \partial^j v/\partial x^j$, $u_0 \equiv u$, $v_0 \equiv v$.

Using the so-called symbolic method Mikhailov *et al.* [MC] proved that the system (22) possesses infinitely many higher symmetries of orders $m \equiv 1, 5 \mod 6$. However, no recursion operator, symplectic or (bi-)Hamiltonian structure for (22) was known.

In [1] we have obtained the following result.

Theorem 5.1. *The system (22) possesses a Hamiltonian operator*

$$
\mathcal{P} = \begin{pmatrix} D_x^3 - \frac{6}{5}uD_x - \frac{3}{5}u_1 & -\frac{6}{5}vD_x - \frac{3}{5}v_1 \\ -\frac{6}{5}vD_x - \frac{3}{5}v_1 & 3D_x^3 - (\frac{18}{5}u + \frac{12}{5}v)D_x - \frac{9}{5}u_1 - \frac{6}{5}v_1 \end{pmatrix},
\tag{23}
$$

a symplectic operator

$$
\mathcal{S} = \begin{pmatrix} S_{11} + \frac{6}{5}\sum\limits_{i=1}^{2}\gamma_{(3-i)1}D_x^{-1}\circ\gamma_{i1} & S_{12} + \frac{6}{5}\sum\limits_{i=1}^{2}\gamma_{(3-i)1}D_x^{-1}\circ\gamma_{i2} \\ S_{21} + \frac{6}{5}\gamma_{22}D_x^{-1}\circ\gamma_{11} & S_{22} + \frac{6}{5}\gamma_{22}D_x^{-1}\circ\gamma_{12} \end{pmatrix},
\tag{24}
$$

and a hereditary recursion operator $\mathcal{R} = \mathcal{P}\circ\mathcal{S}$ that can be written as

$$
\mathcal{R} = \begin{pmatrix} R_{11} + \sum\limits_{i=1}^{2} G_{1i}D_x^{-1}\circ\gamma_{i1} & R_{12} + \sum\limits_{i=1}^{2} G_{1i}D_x^{-1}\circ\gamma_{i2} \\ R_{21} + \sum\limits_{i=1}^{2} G_{2i}D_x^{-1}\circ\gamma_{i1} & R_{22} + \sum\limits_{i=1}^{2} G_{2i}D_x^{-1}\circ\gamma_{i2} \end{pmatrix},
\tag{25}
$$

where

$$S_{11} = -D_x^3 + 6uD_x + 3u_1, \quad S_{12} = -6vD_x + 3v_1,$$

$$S_{21} = -6vD_x - 9v_1, \quad S_{22} = 9D_x^3 - (\tfrac{54}{5}u - \tfrac{36}{5}v)D_x - \tfrac{27}{5}u_1 + \tfrac{18}{5}v_1,$$

$$\gamma_{11} = 1, \quad \gamma_{12} = 0, \quad \gamma_{21} = u_2 - \tfrac{12}{5}u^2 + \tfrac{6}{5}v^2, \quad \gamma_{22} = -\tfrac{6}{5}v^2 + \tfrac{12}{5}uv - 3v_2,$$

$$
\begin{aligned}
R_{11} &= D_x^6 - \tfrac{36}{5}uD_x^4 - \tfrac{108}{5}u_1D_x^3 - (\tfrac{147}{5}u_2 - \tfrac{324}{25}u^2 + \tfrac{252}{25}v^2)D_x^2 \\
&\quad -(21u_3 - \tfrac{216}{5}uu_1 + 36vv_1)D_x - \tfrac{39}{5}u_4 + \tfrac{738}{25}uu_2 - \tfrac{666}{25}vv_2 + \tfrac{621}{25}u_1^2 \\
&\quad -\tfrac{423}{25}v_1^2 - \tfrac{864}{125}u^3 + \tfrac{864}{125}uv^2 - \tfrac{216}{125}v^3, \\
R_{12} &= \tfrac{84}{5}vD_x^4 + \tfrac{102}{5}v_1D_x^3 + (\tfrac{63}{5}v_2 - \tfrac{576}{25}uv + \tfrac{252}{25}v^2)D_x^2 \\
&\quad +(\tfrac{21}{5}v_3 + \tfrac{576}{25}vv_1 - \tfrac{144}{5}vu_1 - \tfrac{396}{25}uv_1)D_x - \tfrac{216}{125}uv^2 + \tfrac{432}{125}u^2v \\
&\quad +\tfrac{36}{5}vv_2 - \tfrac{234}{25}u_2v - \tfrac{18}{5}uv_2 - \tfrac{36}{5}u_1v_1 + \tfrac{126}{25}v_1^2 + \tfrac{3}{5}v_4,
\end{aligned}
$$

$$
\begin{aligned}
R_{21} \;=\;& \tfrac{84}{5}vD_x^4+\tfrac{402}{5}v_1D_x^3-(\tfrac{576}{25}uv-\tfrac{729}{5}v_2+\tfrac{252}{25}v^2)D_x^2\\
&-(\tfrac{648}{25}vu_1+\tfrac{1908}{25}uv_1-\tfrac{657}{5}v_3+\tfrac{432}{25}vv_1)D_x+\tfrac{216}{125}v^2u-\tfrac{1782}{25}uv_2\\
&-\tfrac{108}{25}vv_2-\tfrac{486}{25}vu_2+\tfrac{297}{5}v_4+\tfrac{378}{25}v_1^2+\tfrac{432}{125}u^2v-\tfrac{1656}{25}u_1v_1,\\
R_{22} \;=\;& -27D_x^6+\tfrac{324}{5}uD_x^4+(\tfrac{648}{5}u_1-\tfrac{324}{5}v_1)D_x^3\\
&+(\tfrac{252}{25}v^2-\tfrac{972}{25}u^2-\tfrac{486}{5}v_2+\tfrac{729}{5}u_2)D_x^2\\
&+(81u_3-54v_3-\tfrac{1944}{25}uu_1+\tfrac{684}{25}vv_1-\tfrac{648}{25}vu_1+\tfrac{648}{25}uv_1)D_x-\tfrac{486}{25}uu_2\\
&+\tfrac{432}{125}uv^2-\tfrac{324}{25}vu_2+\tfrac{324}{25}uv_2+\tfrac{198}{25}vv_2-\tfrac{54}{5}v_4+\tfrac{153}{25}v_1^2-\tfrac{243}{25}u_1^2\\
&+\tfrac{81}{5}u_4-\tfrac{216}{125}v^3,\\
G_{11} \;=\;& -\tfrac{6}{5}u_5-\tfrac{36}{5}vv_3-\tfrac{54}{5}v_1v_2+\tfrac{36}{5}uu_3+18u_1u_2-\tfrac{108}{25}v^2v_1+\tfrac{108}{25}v^2u_1\\
&+\tfrac{216}{25}uvv_1-\tfrac{216}{25}u^2u_1,\\
G_{21} \;=\;& \tfrac{54}{5}v_5+\tfrac{108}{5}v_1v_2-\tfrac{108}{5}v_3u-\tfrac{162}{5}v_2u_1-\tfrac{126}{5}v_1u_2-\tfrac{36}{5}vu_3-\tfrac{108}{25}v^2v_1\\
&+\tfrac{108}{25}v^2u_1+\tfrac{216}{25}u^2v_1+\tfrac{216}{25}vuu_1,
\end{aligned}
$$

The recursion operator in the sense of Definition 4.5 can be readily obtained from (25) using the formulas given in (19). For the definition and properties of Hamiltonian and symplectic operators see e.g. [O2, BL].

The operators from Theorem 5.1 were obtained by proceeding in the spirit of the direct approach, see e.g. [MA] for the recursion operators and [KK] for Hamiltonian operators. The idea of the method consists in finding a few low-order symmetries and cosymmetries and a subsequent attempt to construct nonlocal parts of the recursion, symplectic and Hamitonian operators using these quantities.

Denote

$$Q_1=\begin{pmatrix} u_1\\ v_1\end{pmatrix},\quad Q_2=\mathcal{H}\delta\mathcal{P}_0,$$

i.e. Q_2 is a column containing the right-hand sides of (22). The recursion operator (25) and the symmetries with the characteristics Q_1 and Q_2 are readily verified to meet the requirements of Theorem 1 from [S2], and therefore the symmetries with the characteristics $Q_{i,j}=\mathcal{R}^j(Q_i)$, $i=1,2$, $j=0,1,2,\dots$, are higher symmetries for (22) in the sense of Definition 3.1, free of any nonlocal variables. In fact, it can be shown that for any given i and j the characteristic $Q_{i,j}$ depends only on $u,v,u_1,v_1,\dots,u_{1+4(i-1)+6j},v_{1+4(i-1)+6j}$. Moreover, as the recursion operator (25) is hereditary and the symmetries with the characteristics Q_1 and Q_2 commute, so do the symmetries with characteristics $Q_{i,j}$ for all $i=1,2$ and all $j=0,1,2,\dots$

It readily follows from Theorem 5.1 that the system (22) has, as usually is the case for integrable systems (see e.g. [BL]), infinite hierarchies of compatible Hamiltonian operators $\mathcal{R}^k\circ\mathcal{P}$ and symplectic operators $\mathcal{S}\circ\mathcal{R}^k$, $k=0,1,2\dots$ In particular, this means that (22) is a multi-Hamiltonian system.

While the Hamiltonian operator $\mathcal{P}$ is local, it is straightforward to verify that all Hamiltonian operators of the form $\mathcal{R}^k\circ\mathcal{P}$, $k=1,2,\dots$, are nonlocal. We conjecture that $\mathcal{P}$ is the only local Hamiltonian structure for the Mikhailov–Novikov–Wang system (22). Note also that all symplectic structures $\mathcal{S}\circ\mathcal{R}^k$, $k=0,1,2,\dots$, including $\mathcal{S}$ itself, are nonlocal.

Furthermore, it is possible to construct two infinite sequences of conserved functionals $H_{1,k}$ and $H_{2,k}$ given by the formula

$$\delta H_{i,k} = (\mathcal{R}^*)^k(\delta H_i), \tag{26}$$

where

$$H_1 = -\frac{5}{3}\int u dx, \qquad H_2 = \int \left(\frac{5}{6}u_1^2 - \frac{5}{2}v_1^2 + \frac{4}{3}u^3 + \frac{2}{3}v^3 - 2uv^2\right) dx,$$

$\mathcal{R}^* = \mathcal{S} \circ \mathcal{P}$ is the formal adjoint of $\mathcal{R}$ and δ stands for the variational derivative of a functional,

$$\delta \int \rho dx = \sum_{j=0}^{\infty}(-D_x)^j \partial\rho/\partial\vec{u}_j$$

where $\vec{u}_j = (u_j, v_j)^T$, cf. e.g. [O2]. It is readily seen that all functionals $H_{i,k}$ are in involution with respect to Poisson brackets associated with the Hamiltonian structures $\mathcal{R}^s \circ \mathcal{P}$ for all $s = 0, 1, 2, \ldots$. By Proposition 2 of [S2], for all functionals $H_{i,k} \equiv \int \rho_{i,k} dx$, $i = 0, 1, 2, \ldots$, $k = 1, 2$, their densities $\rho_{i,k}$ defined recursively through (26) are local.

As a final remark, note that (22) can be written in the Hamiltonian form as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{P}\delta H_2. \tag{27}$$

6. On nonlocal symmetries for the Krichever–Novikov equation

In the paper [2] we consider the Krichever–Novikov (KN) equation

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}. \tag{28}$$

Here we assume that $P(u) = u^3 + c_1 u + c_0$ is a third-order polynomial in the reduced form (i.e., without quadratic term and with the leading coefficient equal to 1), $c_0, c_1 \in \mathbb{R}$. However, using suitable fractional linear changes of the dependent variable $u \equiv u_{0x}$ we can easily turn (28) into the other known forms of the KN equation with P being a general third- or fourth-degree polynomial in u, cf. [DS].

The KN equation first appeared in [KN] in connection with the study of finite-gap solutions of the Kadomtsev–Petviashvili equation which has plenty of physical applications from plasma physics to fluid dynamics, see e.g. [NP] and references therein.

For the KN equation (28) there exist [DS, SO] two weakly nonlocal recursion operators $\mathcal{R}_1$ and $\mathcal{R}_2$ of orders 4 and 6, respectively, which satisfy [DS] the relation (elliptic curve)

$$\mathcal{R}_2^2 = \mathcal{R}_1^3 - \phi\mathcal{R}_1 - \theta, \tag{29}$$

where

$$\phi = \frac{16}{27}\Big((P'')^2 - 2P'''P' + 2P^{(IV)}P\Big),$$

$$\theta = \frac{128}{243}\Big(-\frac{1}{3}(P'')^3 - \frac{3}{2}(P')^2P^{(IV)} + P'P''P''' + 2P^{(IV)}P''P - P(P''')^2\Big).$$

When written according to Definition 4.5, $\mathcal{R}_1$ is given by (19) with

$$\begin{aligned} V = \mathcal{R}_1(U) = D_x^4(U) + a_1D_x^3(U) + a_2D_x^2(U) + a_3D_x(U) + a_4U \\ + \quad G_1W_1 + u_xW_2, \end{aligned} \tag{30}$$

where G_1 denotes the right-hand side of (28),

$$a_1 = -4\frac{u_{xx}}{u_x}, \qquad a_2 = -2\frac{u_{xxx}}{u_x} + 6\frac{u_{xx}^2}{u_x^2} - \frac{4}{3}\frac{P}{u_x^2},$$

$$a_3 = -2\frac{u_{4x}}{u_x} + 8\frac{u_{xx}u_{xxx}}{u_x^2} - 6\frac{u_{xx}^3}{u_x^3} + 4\frac{u_{xx}}{u_x^3}P - \frac{2}{3}\frac{P'}{u_x},$$

$$\begin{aligned} a_4 = {} & \frac{u_{5x}}{u_x} - 4\frac{u_{xx}u_{4x}}{u_x^2} - 2\frac{u_{xxx}^2}{u_x^2} + 8\frac{u_{xx}^2u_{xxx}}{u_x^3} - 3\frac{u_{xx}^4}{u_x^4} + \frac{4}{9}\frac{P^2}{u_x^4} + \frac{4}{3}\frac{u_{xx}^2}{u_x^4}P \\ & - \frac{8}{3}\frac{u_{xx}}{u_x^2}P' + \frac{10}{9}P'', \end{aligned}$$

and $\gamma_i = \delta\rho_i/\delta u$; here

$$\rho_1 = -\frac{1}{2}\frac{u_{xx}^2}{u_x^2} - \frac{1}{3}\frac{P}{u_x^2}, \qquad \rho_2 = \frac{1}{2}\frac{u_{xxx}^2}{u_x^2} - \frac{3}{8}\frac{u_{xx}^4}{u_x^4} + \frac{5}{6}\frac{u_{xx}^2}{u_x^4}P + \frac{1}{18}\frac{P^2}{u_x^4} - \frac{5}{9}P'',$$

and $\delta/\delta u$ is the variational derivative (cf. e.g. [O2]),

$$\delta\rho/\delta u = \sum_{j=0}^{\infty}(-D_x)^j\partial\rho/\partial u_{jx}.$$

It is obvious (cf. [DS]) that the ratio $\mathcal{R}_3 = \mathcal{R}_2\circ\mathcal{R}_1^{-1}$ is a recursion operator of order two for (28). However, this operator is not weakly nonlocal in the sense of [MN] and, as it was pointed out in [DS], it was unclear how to apply it even to the simplest symmetries of (28), for instance to u_x.

Note that for many equations it is possible to obtain the shadows of nonlocal symmetries by applying their recursion operators to the scaling symmetries, see e.g. [OE]. However, the KN equation (28) has no scaling symmetry, so this approach does not work. One could also try to construct nonlocal variables as potentials for conservation laws and subsequently look for (shadows of) nonlocal symmetries depending on these variables, cf. e.g. [S1] and references therein for more details; however, this method also gave no results for the equation in question. Thus, no nonlocal symmetries (or even shadows thereof) for (28) were known to date.

In [2] we construct new infinite hierarchies of shadows[2] of nonlocal symmetries and cosymmetries for (28) using the inverse $\widetilde{\mathcal{R}}$ of the fourth-order recursion operator $\mathcal{R}_1$.

We also address in [2] the problem of how to apply the recursion operator $\mathcal{R}_3$ to the known symmetries of (28).

In order to recall the results of [2], introduce the nonlocal variables p_i, q_i, z_i, $i = 1, 2$, defined by the following relations (see Appendix of [2] for the motivation of this definition):

$$
\begin{aligned}
&(p_1)_x = k_3 p_1^2 + 2k_1 p_1 - k_2, && (p_1)_t = l_3 p_1^2 + 2l_1 p_1 - l_2,\\
&(z_1)_x = -(k_1 + p_1 k_3), && (z_1)_t = -(l_1 + p_1 l_3),\\
&(q_1)_x = -k_3 \exp(-2z_1), && (q_1)_t = -l_3 \exp(-2z_1),\\
&(p_2)_x = -k_3 p_2^2 - 2k_1 p_2 + k_2, && (p_2)_t = -\left(l_3 - m\right) p_2^2\\
& && \qquad - \left(\frac{4u^2}{3u_x} + 2l_1\right) p_2 - \frac{2c_1}{3u_x} + l_2,\\
&(z_2)_x = (k_1 + p_2 k_3), && (z_2)_t = l_1 + \frac{2u^2}{3u_x} + p_2\left(l_3 - m\right),\\
&(q_2)_x = k_3 \exp(-2z_2), && (q_2)_t = (l_3 - m)\exp(-2z_2).
\end{aligned}
\tag{31}
$$

Here

$$
\begin{aligned}
k_1 &= -\frac{\sqrt{6}(c_1 u + 2c_0)}{12\sqrt{c_0}u_x}, \qquad k_2 = \frac{\sqrt{6}c_1 u}{12\sqrt{c_0}u_x},\\
k_3 &= \frac{\sqrt{6}u(4c_0 u - c_1^2)}{12c_1\sqrt{c_0}u_x}, \qquad m = \frac{2(c_1^2 - 8c_0 u - 2c_1 u^2)}{3c_1 u_x},\\
l_1 &= -\frac{\sqrt{6}}{72\sqrt{c_0}u_x^3}(-6c_1 u u_x u_{xxx} - 12c_0 u_x u_{xxx} + 3c_1 u u_{xx}^2 + 6c_0 u_{xx}^2\\
&\quad + 12c_1 u_x^2 u_{xx} + 4\sqrt{6}\sqrt{c_0}u^2 u_x^2 - 2c_1 u^4 - 4c_0 u^3 - 2c_1^2 u^2\\
&\quad - 6c_0 c_1 u - 4c_0^2),\\
l_2 &= \frac{\sqrt{6}c_1}{72\sqrt{c_0}u_x^3}(-6u u_x u_{xxx} + 3u u_{xx}^2 + 12u_x^2 u_{xx} + 4\sqrt{6}\sqrt{c_0}u_x^2 - 2u^4\\
&\quad - 2c_1 u^2 - 2c_0 u),\\
l_3 &= -\frac{\sqrt{6}}{72c_1\sqrt{c_0}u_x^3}(-6c_1^2 u u_x u_{xxx} + 24c_0 u^2 u_x u_{xxx} + 3c_1^2 u u_{xx}^2\\
&\quad - 12c_0 u^2 u_{xx}^2 + 12c_1^2 u_x^2 u_{xx} - 96c_0 u u_x^2 u_{xx} + 96c_0 u_x^4 - 4\sqrt{6}\sqrt{c_0}c_1^2 u_x^2\\
&\quad + 32\sqrt{6}c_0^{\frac{3}{2}} u u_x^2 + 8\sqrt{6}\sqrt{c_0}c_1 u^2 u_x^2 + 8c_0 u^5 - 2c_1^2 u^4 + 8c_0 c_1 u^3\\
&\quad - 2c_1^3 u^2 + 8c_0^2 u^2 - 2c_0 c_1^2 u).
\end{aligned}
\tag{32}
$$

[2]The terminology used in [2] is slightly different in that in order to streamline the presentation the shadows were referred to as nonlocal (co)symmetries, as it is often done in the literature (cf. e.g. [BL, OE]).

Geometrically, equations (31) define a six-dimensional covering over (28). Now define the quantities V_i and γ_i, $i = 1, 2, \ldots, 6$, as follows:

$$V_1 = \frac{c_1 u + 2c_0}{2c_0} + \frac{1}{4c_1c_0} \sum_{i=1}^{2} \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i \right.$$
$$\left. + c_1^2 u)(q_i - 1)\exp(2z_i) + u(c_1^2 - 4c_0u)p_i\right],$$

$$V_2 = -\frac{1}{2c_1^2} \sum_{i=1}^{2} ((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)\exp(2z_i),$$

$$V_3 = \frac{\sqrt{6}}{8c_1^2\sqrt{c_0}} \sum_{i=1}^{2} (-1)^{i-1}((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)\exp(2z_i),$$

$$V_4 = -\frac{\sqrt{6}}{64\sqrt{c_0^3}} \sum_{i=1}^{2} (-1)^{i-1} \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i \right.$$
$$+ c_1^2u)(q_i - 1)^2\exp(2z_i) + u(c_1^2 - 4c_0u)\exp(-2z_i)$$
$$\left. + 2u(c_1^2 - 4c_0u)p_i(q_i - 1) + 2c_1(c_1u + 2c_0)q_i\right],$$

$$V_5 = -\frac{1}{32c_0} \sum_{i=1}^{2} \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)(q_i^2 - 1)\exp(2z_i) \right.$$
$$\left. + u(c_1^2 - 4c_0u)\exp(-2z_i) + 2u(c_1^2 - 4c_0u)p_iq_i + 2c_1(c_1u + 2c_0)q_i\right],$$

$$V_6 = \frac{\sqrt{6}}{16c_1\sqrt{c_0}} \sum_{i=1}^{2} (-1)^{i-1} \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i \right.$$
$$\left. + c_1^2u)q_i\exp(2z_i) + u(c_1^2 - 4c_0u)p_i\right],$$

$$\gamma_1 = \frac{\sqrt{6}}{16c_1\sqrt{c_0}u_x^3} \sum_{i=1}^{2} (-1)^{i-1} \left\{ \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)u_{xx} \right.\right.$$
$$\left. + ((8c_0u - c_1^2)p_i^2 - 2c_1^2p_i - c_1^2)u_x^2\right] q_i\exp(2z_i)$$
$$\left. + \left[(8c_0u - c_1^2)u_x^2 + (c_1^2 - 4c_0u)uu_{xx}\right] p_i \right\},$$

$$\gamma_2 = \frac{\sqrt{6}}{64\sqrt{c_0^3}u_x^3} \sum_{i=1}^{2} (-1)^{i-1} \left\{ \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)u_{xx} \right.\right.$$
$$\left. + ((8c_0u - c_1^2)p_i^2 - 2c_1^2p_i - c_1^2)u_x^2\right] (q_i - 1)^2\exp(2z_i)$$
$$+ \left[(8c_0u - c_1^2)u_x^2 + (c_1^2 - 4c_0u)uu_{xx}\right]\exp(-2z_i) + 2\left[(8c_0u - c_1^2)u_x^2\right.$$
$$\left.\left. + (c_1^2 - 4c_0u)uu_{xx}\right] p_i(q_i - 1) + 2c_1\left[(2c_0 + c_1u)u_{xx} - c_1u_x^2\right] q_i \right\},$$

$$\gamma_3 = -\frac{1}{16c_0u_x^3} \sum_{i=1}^{2} \left\{ \left[((c_1^2 - 4c_0u)up_i^2 + 2c_1(c_1u + 2c_0)p_i + c_1^2u)u_{xx} \right.\right.$$
$$\left. + ((8c_0u - c_1^2)p_i^2 - 2c_1^2p_i - c_1^2)u_x^2\right] (q_i^2 - 1)\exp(2z_i)$$
$$+ \left[(8c_0u - c_1^2)u_x^2 + (c_1^2 - 4c_0u)uu_{xx}\right]\exp(-2z_i) + 2\left[(8c_0u - c_1^2)u_x^2\right.$$
$$\left.\left. + (c_1^2 - 4c_0u)uu_{xx}\right] p_iq_i + 2c_1\left[(2c_0 + c_1u)u_{xx} - c_1u_x^2\right] q_i \right\},$$

$$\gamma_4 = \frac{1}{2c_1^2u_x^3}\sum_{i=1}^{2}\left[((c_1^2-4c_0u)up_i^2+2c_1(c_1u+2c_0)p_i+c_1^2u)u_{xx}\right.$$
$$\left.+((8c_0u-c_1^2)p_i^2-2c_1^2p_i-c_1^2)u_x^2\right]\exp(2z_i),$$
$$\gamma_5 = \frac{\sqrt{6}}{4c_1^2\sqrt{c_0}u_x^3}\sum_{i=1}^{2}(-1)^{i-1}\left[((c_1^2-4c_0u)up_i^2+2c_1(c_1u+2c_0)p_i+c_1^2u)u_{xx}\right.$$
$$\left.+((8c_0u-c_1^2)p_i^2-2c_1^2p_i-c_1^2)u_x^2\right]\exp(2z_i),$$
$$\gamma_6 = \frac{(2c_0+c_1u)u_{xx}-c_1u_x^2}{2c_0u_x^3}+\frac{1}{4c_1c_0u_x^3}\sum_{i=1}^{2}\left\{\left[((c_1^2-4c_0u)up_i^2+2c_1(c_1u+2c_0)p_i\right.\right.$$
$$\left.+c_1^2u)u_{xx}+((8c_0u-c_1^2)p_i^2-2c_1^2p_i-c_1^2)u_x^2\right](q_i-1)\exp(2z_i)$$
$$\left.+\left[(8c_0u-c_1^2)u_x^2+(c_1^2-4c_0u)uu_{xx}\right]p_i\right\}.$$

It turns out that the following assertions hold [2].

Proposition 6.1. *The quantities V_i (resp. γ_i), $i = 1, \ldots, 6$, are shadows of nonlocal symmetries (resp. nonlocal cosymmetries) for the KN equation (28) with respect to the covering (31).*

Theorem 6.2. *The KN equation (28) possesses a recursion operator $\widetilde{\mathcal{R}}$ whose action on a symmetry U (or, more broadly, on a shadow of nonlocal symmetry) of (28) is given by the formula*

$$\widetilde{\mathcal{R}}(U) = (S^{-1}\vec{\Omega})_1 \equiv \sum_{k=1}^{6}(S^{-1})_{1k}\Omega_k, \tag{33}$$

where $\vec{\Omega}$ is a vector of nonlocal variables defined by the relations

$$\vec{\Omega}_x = SM_1U, \qquad \vec{\Omega}_t = SM_2U. \tag{34}$$

Furthermore, $\widetilde{\mathcal{R}}$ is the inverse of the fourth-order recursion operator $\mathcal{R}_1$ written in the form (30), that is, modulo arbitrary integration constants arising from the definition of the nonlocal variables W_i and $\vec{\Omega}$ we have that

$$\widetilde{\mathcal{R}}(\mathcal{R}_1(U)) = \mathcal{R}_1(\widetilde{\mathcal{R}}(U)) = U.$$

The matrices M_1 and M_2 in Theorem 6.2 have the forms

$$M_1 = \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0\\ 1\\ D_x+\dfrac{u_{xx}}{u_x}\\ D_x^2+\dfrac{u_{xx}}{u_x}D_x+\dfrac{6u_xu_{xxx}+3u_{xx}^2+2P}{6u_x^2}\\ 0\\ 0 \end{pmatrix},$$

while the matrix S is given by the formula

$$S = \exp(q_2Y_2)\cdot\exp(z_2H_2)\cdot\exp(p_2X_2)\cdot\exp(q_1Y_1)\cdot\exp(z_1H_1)\cdot\exp(p_1X_1)\cdot S^{(0)},$$

where

$$X_1 = \begin{pmatrix} 1 & \frac{2c_0}{c_1} & -\frac{\sqrt{6c_0}}{2c_1} & \frac{\sqrt{6}c_1}{16\sqrt{c_0}} & -\frac{c_1}{8} & 0 \\ -\frac{c_1}{4c_0} & -1 & \frac{\sqrt{6}}{4\sqrt{c_0}} & 0 & \frac{c_1^2}{16c_0} & \frac{\sqrt{6}c_1}{16\sqrt{c_0}} \\ \frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 & 0 & \frac{c_1^2}{8c_0} & 0 & \frac{c_1}{4} \\ \frac{4\sqrt{6c_0}}{3c_1} & 0 & 0 & 1 & 0 & \frac{2c_0}{c_1} \\ \frac{4}{c_1} & 0 & 0 & \frac{\sqrt{6}}{2\sqrt{c_0}} & 0 & \frac{\sqrt{6c_0}}{c_1} \\ 0 & \frac{4\sqrt{6c_0}}{3c_1} & -\frac{2}{c_1} & -\frac{c_1}{4c_0} & -\frac{\sqrt{6}c_1}{12\sqrt{c_0}} & -1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{\sqrt{6}c_1}{16\sqrt{c_0}} & \frac{c_1}{8} & 0 \\ \frac{c_1}{4c_0} & 0 & 0 & 0 & -\frac{c_1^2}{16c_0} & \frac{\sqrt{6}c_1}{16\sqrt{c_0}} \\ -\frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 & 0 & -\frac{c_1^2}{8c_0} & 0 & -\frac{c_1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_1}{4c_0} & \frac{\sqrt{6}c_1}{12\sqrt{c_0}} & 0 \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{c_1}{2c_0} & -1 & \frac{\sqrt{6}}{4\sqrt{c_0}} & 0 & \frac{c_1^2}{8c_0} & 0 \\ \frac{\sqrt{6}c_1}{3\sqrt{c_0}} & \frac{2\sqrt{6c_0}}{3} & -1 & \frac{c_1^2}{4c_0} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\sqrt{6c_0}}{3} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{6}}{2\sqrt{c_0}} & 1 & 0 \\ 0 & 0 & 0 & -\frac{c_1}{2c_0} & -\frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 1 & \frac{2c_0}{c_1} & \frac{\sqrt{6c_0}}{2c_1} & -\frac{\sqrt{6}c_1}{16\sqrt{c_0}} & -\frac{c_1}{8} & 0 \\ -\frac{c_1}{4c_0} & -1 & -\frac{\sqrt{6}}{4\sqrt{c_0}} & 0 & \frac{c_1^2}{16c_0} & -\frac{\sqrt{6}c_1}{16\sqrt{c_0}} \\ -\frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 & 0 & \frac{c_1^2}{8c_0} & 0 & \frac{c_1}{4} \\ -\frac{4\sqrt{6c_0}}{3c_1} & 0 & 0 & 1 & 0 & \frac{2c_0}{c_1} \\ \frac{4}{c_1} & 0 & 0 & -\frac{\sqrt{6}}{2\sqrt{c_0}} & 0 & -\frac{\sqrt{6c_0}}{c_1} \\ 0 & -\frac{4\sqrt{6c_0}}{3c_1} & -\frac{2}{c_1} & -\frac{c_1}{4c_0} & \frac{\sqrt{6}c_1}{12\sqrt{c_0}} & -1 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{\sqrt{6}c_1}{16\sqrt{c_0}} & \frac{c_1}{8} & 0 \\ \frac{c_1}{4c_0} & 0 & 0 & 0 & -\frac{c_1^2}{16c_0} & -\frac{\sqrt{6}c_1}{16\sqrt{c_0}} \\ \frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 & 0 & -\frac{c_1^2}{8c_0} & 0 & -\frac{c_1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_1}{4c_0} & -\frac{\sqrt{6}c_1}{12\sqrt{c_0}} & 0 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{c_1}{2c_0} & -1 & -\frac{\sqrt{6}}{4\sqrt{c_0}} & 0 & \frac{c_1^2}{8c_0} & 0 \\ -\frac{\sqrt{6}c_1}{3\sqrt{c_0}} & -\frac{2\sqrt{6c_0}}{3} & -1 & \frac{c_1^2}{4c_0} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\sqrt{6c_0}}{3} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{6}}{2\sqrt{c_0}} & 1 & 0 \\ 0 & 0 & 0 & -\frac{c_1}{2c_0} & \frac{\sqrt{6}c_1}{6\sqrt{c_0}} & 0 \end{pmatrix},$$

and

$$S^{(0)} = \begin{pmatrix} S^{(0)}_{11} & -\dfrac{u}{u_x} & \dfrac{1}{4}\dfrac{u^2}{u_x^2} & 0 & -\dfrac{1}{4}\dfrac{u^2}{u_x} & 0 \\ S^{(0)}_{21} & -\dfrac{1}{u_x} & \dfrac{1}{2}\dfrac{u}{u_x^2} & 0 & -\dfrac{1}{2}\dfrac{u}{u_x} & 0 \\ S^{(0)}_{31} & S^{(0)}_{32} & S^{(0)}_{33} & \dfrac{1}{2}\dfrac{u^2u_{xx}}{u_x^3} - \dfrac{u}{u_x} & S^{(0)}_{35} & -\dfrac{1}{2}\dfrac{u^2}{u_x} \\ S^{(0)}_{41} & S^{(0)}_{42} & S^{(0)}_{43} & \dfrac{uu_{xx}}{u_x^3} - \dfrac{1}{u_x} & S^{(0)}_{45} & -\dfrac{u}{u_x} \\ S^{(0)}_{51} & 0 & \dfrac{1}{u_x^2} & 0 & -\dfrac{1}{u_x} & 0 \\ S^{(0)}_{61} & S^{(0)}_{62} & -\dfrac{u_{xxx}}{u_x^3} - \dfrac{u_{xx}^2}{u_x^4} & \dfrac{u_{xx}}{u_x^3} & \dfrac{1}{2}\dfrac{u_{xx}^2}{u_x^3} - \dfrac{1}{3u_x^3}P & -\dfrac{1}{u_x} \end{pmatrix},$$

where

$$S^{(0)}_{11} = -\frac{1}{4}\frac{u^2u_{xxx}}{u_x^3}+\frac{1}{4}\frac{u^2u_{xx}^2}{u_x^4}-\frac{1}{6}\frac{u^2}{u_x^4}P+1, \quad S^{(0)}_{21} = -\frac{1}{2}\frac{uu_{xxx}}{u_x^3}+\frac{1}{2}\frac{uu_{xx}^2}{u_x^4}-\frac{1}{3}\frac{u}{u_x^4}P,$$

$$\begin{aligned} S^{(0)}_{31} &= \frac{2u^2u_xu_{xxx} - 4u^2u_{xx}^2 - 2uu_x^2u_{xx} + 2u_x^4}{3u_x^6}P + \frac{1}{6}\frac{u(5uu_{xx} - 2u_x^2)}{u_x^4}P' \\ &- \frac{5}{18}\frac{u^2}{u_x^2}P'' - \frac{1}{2}\frac{u^2u_{5x}}{u_x^3} + 2\frac{u^2u_{xx}u_{4x}}{u_x^4} + \frac{3}{2}\frac{u^2u_{xxx}^2}{u_x^4} \\ &- \frac{(5u^2u_{xx}^2 + uu_x^2u_{xx} - u_x^4)u_{xxx}}{u_x^5} + \frac{(uu_{xx} + u_x^2)(2uu_{xx} - u_x^2)u_{xx}^2}{u_x^6}, \end{aligned}$$

$$\begin{aligned} S^{(0)}_{32} &= -\frac{2u(uu_{xx} - u_x^2)}{3u_x^5}P + \frac{u^2}{6u_x^3}P' + \frac{1}{2}\frac{u^2u_{4x}}{u_x^3} + \frac{1}{2}\frac{(-3uu_{xx} + 2u_x^2)uu_{xxx}}{u_x^4} \\ &- \frac{(uu_{xx} + u_x^2)(-uu_{xx} + 2u_x^2)u_{xx}}{u_x^5}, \end{aligned}$$

$$S^{(0)}_{33} = -\frac{1}{2}\frac{u^2u_{xxx}}{u_x^3}-\frac{1}{2}\frac{u^2u_{xx}^2}{u_x^4}+2\frac{uu_{xx}}{u_x^2}+1, \quad S^{(0)}_{35} = \frac{1}{4}\frac{u^2u_{xx}^2}{u_x^3}-\frac{uu_{xx}}{u_x}+u_x-\frac{1}{6}\frac{Pu^2}{u_x^3},$$

$$\begin{aligned} S^{(0)}_{41} &= \frac{4uu_xu_{xxx} - 8uu_{xx}^2 - 2u_x^2u_{xx}}{3u_x^6}P + \frac{5uu_{xx} - u_x^2}{3u_x^4}P' - \frac{5u}{9u_x^2}P'' - \frac{uu_{5x}}{u_x^3} \\ &+ 4\frac{uu_{xx}u_{4x}}{u_x^4} + 3\frac{uu_{xxx}^2}{u_x^4} - \frac{(10uu_{xx} + u_x^2)u_{xx}u_{xxx}}{u_x^5} + \frac{(4uu_{xx} + u_x^2)u_{xx}^3}{u_x^6}, \end{aligned}$$

$$\begin{aligned} S^{(0)}_{42} &= \frac{1}{3}\frac{-4uu_{xx} + 2u_x^2}{u_x^5}P + \frac{1}{3}\frac{u}{u_x^3}P' + \frac{uu_{4x}}{u_x^3} + \frac{(-3uu_{xx} + u_x^2)u_{xxx}}{u_x^4} \\ &- \frac{(-2uu_{xx} + u_x^2)u_{xx}^2}{u_x^5}, \end{aligned}$$

$$S^{(0)}_{43} = -\frac{uu_{xxx}}{u_x^3} - \frac{uu_{xx}^2}{u_x^4} + 2\frac{u_{xx}}{u_x^2}, \quad S^{(0)}_{45} = \frac{uu_{xx}^2}{2u_x^3} - \frac{u_{xx}}{u_x} - \frac{u}{3u_x^3}P,$$

$$S_{51}^{(0)} = -\frac{u_{xxx}}{u_x^3} + \frac{u_{xx}^2}{u_x^4} - \frac{2}{3u_x^4}P,$$

$$S_{61}^{(0)} = \frac{4u_x u_{xxx} - 8u_{xx}^2}{3u_x^6}P + \frac{5}{3}\frac{u_{xx}}{u_x^4}P' - \frac{5}{9u_x^2}P'' - \frac{u_{5x}}{u_x^3} + 4\frac{u_{xx}u_{4x}}{u_x^4} + 3\frac{u_{xxx}^2}{u_x^4} - 10\frac{u_{xx}^2 u_{xxx}}{u_x^5} + 4\frac{u_{xx}^4}{u_x^6},$$

$$S_{62}^{(0)} = -\frac{4}{3}\frac{u_{xx}}{u_x^5}P + \frac{1}{3u_x^3}P' + \frac{u_{4x}}{u_x^3} - 3\frac{u_{xx}u_{xxx}}{u_x^4} + 2\frac{u_{xx}^3}{u_x^5}.$$

Let us also mention that $\widetilde{\mathcal{R}}$ can be formally written in the pseudodifferential form (cf. e.g. [DS, O2]) as

$$\widetilde{\mathcal{R}} = \sum_{i=1}^{6} V_i \ D_x^{-1} \circ \gamma_i, \tag{35}$$

where V_i and γ_i are shadows of nonlocal symmetries and cosymmetries for (28), see above.

Using Theorem 6.2 we can enhance the result of Proposition 6.1 as follows:

Proposition 6.3. *The quantities* $V_i^{(j)} = \widetilde{\mathcal{R}}^j(V_i)$ *(resp.* $(\widetilde{\mathcal{R}}^\star)^j(\gamma_i)$*),* $i = 1,\dots,6$*, are shadows of nonlocal symmetries (resp. of nonlocal cosymmetries) for the KN equation (28) for all* $j = 0,1,2,\dots$*.*

Here $\widetilde{\mathcal{R}}^\star$ is the formal adjoint of $\widetilde{\mathcal{R}}$; in the pseudodifferential form we can write $\widetilde{\mathcal{R}}^\star = -\sum_{i=1}^6 \gamma_i \ D_x^{-1} \circ V_i$; the correct definition of such an operator is given in analogy with Definition 4.5 with $V\mathcal{E}^\infty$ replaced by the dual object, the so-called cotangent covering over $\mathcal{E}$, see e.g. [KV] for details.

As a final remark, note that it is possible to construct two hierarchies of highly nonlocal Hamiltonian structures for (28) of the form $\widetilde{\mathcal{R}}^j \circ \mathcal{H}_i$, $j = 1,2,\dots$, $i = 0,2$, where $\mathcal{H}_0 = u_x D_x^{-1} \circ u_x$ and $\mathcal{H}_2 = \mathcal{R}_2 \circ \mathcal{H}_0$, cf. [DS, SO].

Publications constituting the thesis

[1] Vojčák P., On complete integrability of the Mikhailov–Novikov–Wang system, *J. Math. Phys.* **52** (2011), 043513.

[2] Vojčák P., On nonlocal symmetries for the Krichever–Novikov equation, *Phys. Lett. A* **376** (2012), 1335-1343.

Presentations related to the thesis

[3] Humboldt Kolleg/XIV International Conference *Symmetry Methods in Physics*, August 16-22, 2010, Tsakhkadzor, Armenia. Talk *Recursion operator, Hamiltonian and symplectic structures for the Mikhailov–Novikov–Wang system.*

[4] The 11th Conference *Mathematics in Technical and Natural Sciences*, September 8-12, 2010, Krynica, Poland. Talk *Recursion operator, Hamiltonian and symplectic structures for the Mikhailov–Novikov–Wang system.*

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