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Various approaches to the blowing-up orbits technique

Abstract of the Ph.D. Thesis
July 2018

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav v Opavě

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Různé přístupy k technice nafukování orbit

Autoreferát dizertační práce
Červenec 2018

Matematická analýza

Výsledky uvedené v této dizertační práci byly získány v průběhu mého doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2011–2018.

Výzkum byl podporován granty SGS/15/2010, SGS/2/2013 a SGS/18/2016.

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Autoreferát této dizertační práce byl zveřejněn dne 13. 11. 2018.

Státní doktorská zkouška a obhajoba dizertační práce se bude konat dne 13. 12. 2018 od 11 hodin před oborovou komisí doktorského studia Matematické analýzy v zasedací místnosti rektorátu Slezské univerzity v Opavě.

Dizertační práce je dostupná v knihovně Matematického ústavu, Na Rybníčku 626/1, 746 01, Opava.

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The author is immensely thankful to his supervisor doc. RNDr. Marta Štefánková, Ph. D. as well as his co-author prof. Sergei Trofimchuk, Ph. D. for their invaluable guidance and support during his study.

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1. INTRODUCTION

This thesis is based on two independent papers, [i] and [ii]. There are two links between them. Firstly, in both of them the main result is a counterexample disproving a conjecture and secondly, both the counterexamples are constructed by the so called blowing-up orbits technique. On the other hand, each of the papers takes a different approach to this common technique and they also focus on different kinds of dynamical systems – one with autonomous ones and the second one with nonautonomous ones.

This abstract consists of four chapters (including this Introduction) – the second chapter contains necessary theoretical and background, the third and fourth chapter outline the construction of the counterexamples. The fourth chapter also contains some minor unpublished results concerning one of the counterexamples.

2. PRELIMINARIES

In this chapter we sum up the basic notions and definitions needed to understand the thesis. By an *autonomous discrete dynamical system* (or ADS) we mean a pair (X, f) of a topological space X and a continuous map f mapping the space X into itself. By a *nonautonomous discrete dynamical system* (or NDS) we mean a pair $(X, f_{1,\infty})$, where $f_{1,\infty}$ is a shorthand for a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous maps from X to itself. In general, X can be arbitrary topological space, but in this thesis we will restrict ourselves to X being compact metric space. Moreover, by I we will denote a compact real interval or, without loss on generality, the interval $[0, 1]$ and $C(X)$ will denote the set of all continuous mappings from X into itself.

Unless stated otherwise, we will define the notions only for NDS as they can be easily transformed to the ADS ones by putting $f_n = f$ for every n. Where necessary, we will comment on the differences. By $f_k^n(x)$ we mean the n-th *iteration* of a point x starting with the map f_k and going forward. More precisely

$$f_k^0(x) = x \tag{2.1}$$

$$f_k^1(x) = f_k(x) \tag{2.2}$$

$$f_k^n(x) = (f_{k+n-1} \circ f_k^{n-1})(x) \tag{2.3}$$

Analogously, we define and denote the n-th iteration of a map. In the ADS case notation we omit the subscript and we can also extend the order of the iteration to the negative values. Although if we do not require the map f to be a homeomorphism than the negative itarations may be larger sets and not only singletons.

The set of all (positive) iterations of a point is called *(forward) orbit*. The word *forward* is put into parentheses as we will omit it where it is not needed. For the autonomous case we can define also *backward orbit*, i.e. a set (or union) of negative iterations of a point. A union of forward and backward orbit is called *full orbit*. Obviously, the backward and full orbit can not be defined for the NDS.

By a *(forward)/backward/full trajectory* of a point we mean the sequence of positive/negative/all iterations of a point. Again, the backward and full trajectory can not be defined for NDS and if we do not require the map f in the ADS to be a homeomorphism the backward and full trajectory of a point may not be uniquely (and thus well) defined. An *ω-limit set of a point x* is the set of the limit points of the orbit of the point. These definitions can be easily extended, e.g., from the orbit of a point to the orbit of a set etc.

Before we get to the next notion – Li-Yorke chaos – we will briefly review its history. It was first defined by Li and Yorke in their paper Period three implies chaos, [18], for ADS. As discussed above, we will present here the definition for NDS, which can be find e.g. in [7].

Definition 1. *Let $(X, f_{1,\infty})$ be a NDS and ϱ the corresponding metric on X. Two distinct points $x,\ y \in X$ form a* Li-Yorke pair *(shortly LY-pair) if and only if it holds:*

$$\limsup_{n\to\infty} \varrho(f_1^n(x), f_1^n(y)) > 0$$
$$\liminf_{n\to\infty} \varrho(f_1^n(x), f_1^n(y)) = 0$$

A set where arbitrary two distinct points form a LY-pair is called Li-Yorke scrambled set. *A NDS $(X, f_{1,\infty})$ is called* Li-Yorke chaotic *(LYC), if X contains an uncountable Li-Yorke scrambled set.*

The next notion we will need is the topological entropy which was introduced by Adler, Konheim and MacAndrew, [1], and later equivalently redefined by Bowen, [5], and the topological sequence entropy, which was (for ADS) introduced by Goodman, [13]. The notion of topological entropy was generalized for NDS by Kolyada and Snoha in [16]. We will state here the natural generalization of their definition to topological sequence entropy for NDS. It is stated in Bowen form since it is more appropriate for our purposes.

Definition 2. *Let X be a compact metric space with metric ϱ and $f_{1,\infty}$ a sequence of maps from $C(X)$. Let $A = \{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. For each positive integer n we put*

$$\varrho_n^A(x, y) := \max_{0\le j\le n-1} \varrho(f_1^{a_j}(x), f_1^{a_j}(y)).$$

A subset E of the space X is then called (n, ε, A)-separated *if for any two distinct points x and y from E it holds $\varrho_n^A(x, y) > \varepsilon$. We denote the maximal cardinality of an (n, ε, A)-separated set in X by $s_n^A(f_{1,\infty}, \varepsilon)$. By a* topological sequence entropy *of $f_{1,\infty}$ with respect to the sequence A we mean*

$$h_A(f_{1,\infty}) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{a_n} \cdot \log s_n^A(f_{1,\infty}, \varepsilon)$$

If we take, in the above definition, $a_i = i$, we get the notion of *topological entropy* of a NDS. And similarly, we can specialize the above definition to topological sequence entropy and topological entropy of ADS.

This concludes the part of this chapter where we needed the NDS. From now on all of the definitions will take into account the ADS only. First, let us shortly take f such that $h(f) = 0$. Then for every infinite ω-limit set $\tilde{\omega}$ there exists an *associated system* $\mathcal{I}(\tilde{\omega}) := \{J(k, n) | 0 \le k < 2^n, n \ge 0\}$ (see [22]) of minimal f-periodic compact intervals of period 2^n and that $J(n + 1, 0) \subset J(n, 0)$ and

$$f(M(\tilde{\omega})) = M(\tilde{\omega}) := \bigcap_{n\ge 0} \bigcup_{0\le k<2^n} J(n, k) \supseteq \tilde{\omega}. \qquad (2.4)$$

The set $M(\tilde{\omega})$ is called *the simple set of the associated system*.

We will also use some notions from symbolic dynamics. By Σ we will denote the *shift space* which is a set of all infinite sequences of two symbols, say 0 and 1. This space is equipped with the lexicographical order and the order topology. Let n_k denote a finite sequence of length k, $n_k \in \{0,1\}^k$ or a k-*block*. Then we denote by Σ_{n_k} an n_k-*cylinder*, i.e., the subset of Σ consisting of those sequences starting with the block n_k. The block n_k is in this context called a *code* of the cylinder Σ_{n_k}.

Let $\underline{n} \in \Sigma$ and $k \in \mathbb{N}$, then by $\underline{n}|k$ we mean a k-block consisting of k first symbols of $\underline{n}$. We also use $\underline{0}$ and $\underline{1}$ as a shorthand for constant sequences consisting of 0's or 1's, respectively. And by $n_k\underline{0}$ we mean a concatenation of the k-block and the infinite sequence $\underline{0}$.

An important role in [i] plays the adding machine $\alpha\colon \Sigma \to \Sigma$. This mapping adds a sequence $1\underline{0}$ to a mapped sequence modulo 2. E.g. a (piece of) the full trajectory of $\underline{0}$ looks like this:

$$\ldots \mapsto 00\underline{1} \mapsto 10\underline{1} \mapsto 0\underline{1} \mapsto \underline{1} \mapsto \underline{0} \mapsto 1\underline{0} \mapsto 01\underline{0} \mapsto 11\underline{0} \mapsto \ldots \tag{2.5}$$

A mapping $\varphi\colon \Sigma \to \Sigma$ is called a *simple map* if for any $\underline{n} \in \Sigma$ and any $k \in \mathbb{N}$ it holds $\varphi^{2^k}(\Sigma_{\underline{n}|k}) = \Sigma_{\underline{n}|k}$ (or the cylinder $\Sigma_{\underline{n}|k}$ is 2^k periodic). It was proved in [6] that a map is simple if and only if it is conjugate to the adding machine.

Finally, notice that we will also use the word *block* for a finite sequence of maps. We believe it will always be obvious from the context whether we mean a block of 0's and 1's or of mappings.

Next notion to introduce is the minimality of an ADS. We call an ADS (X, f) *minimal* if and only if there is no proper subset $X' \subset X$ such that $f(X') = X'$.

Let us now sum up few simple notations. By $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ we mean a circle or a one-dimensional torus. The positive orientation of this manifold is induced by the usual order on $[0,1)$ and by $[x,y]$ we will understand – apart from closed interval – an arc on $\mathbb{T}^1$ where we move from x to y in the positive direction. We believe the reader will be able to distinguish the interval from an arc. Next, we define $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$ – the two-dimensional torus. And finally, by $\mathbb{K}^2$ we denote the Klein bottle. More precisely $\mathbb{K}^2$ is the quotient space of $\mathbb{T}^2$ induced by the following equivalence: $(x,y) \sim (u,v)$ if and only if $P(x,y) = (u,v)$ where $P(x,y) = (x + 1/2, 1 - y)$. The equivalence $\sim$ is well-defined because P is idempotent, i.e. $P^2 = P$.

Last but not least, we will briefly describe the general idea behind *the blowing-up orbits technique*. The objective of this technique is to construct a topological extension of some map, say g and f, respectively. The extension g keeps certain properties of f, but loses others and that is the reason why it is used to construct counterexamples. In general, we take an orbit of a point or a set, say B and define a set-valued map h which assigns certain sets to the points of B and it is then continuously (usually linearly) and surjectively extended to the whole domain of f. The inverse h^{-1} of this set-valued can be interpreted as a point-valued map and then we put $g \circ h^{-1} = h^{-1} \circ f$. Let us conclude by a remark that such maps f and g are said to be *semiconjugate by* h^{-1}, map f is called *base* and g is called *extension* as already used above.

3. RELATIONSHIP OF LI-YORKE CHAOS AND TOPOLOGICAL SEQUENCE ENTROPY

In this chapter we briefly review the content of the paper *Relationship between Li-Yorke chaos and positive topological sequence entropy in nonautonomous dynamical systems*, [i].

3.1 Historical background

In this section, we will discuss the motivation for [i] and recall its historical background.

Among other notions, Li-Yorke chaos and topological entropy belong to basic and widely used concepts in the theory of discrete dynamical systems. Hence, several authors investigated their mutual relationship. Since 2002, it is known (see [3]) that for continuous mappings of compact metric spaces (i. e. ADS) positive topological entropy implies Li-Yorke chaos. However, the converse implication does not hold, [22]. Later, it has been proved in [12] that for maps on the compact interval existence of a sequence such that the corresponding topological sequence entropy is positive is equivalent to Li-Yorke chaos.

The first paper in the field of nonautonomous discrete dynamical systems is due to Kolyada and Snoha, [16]. In the paper, the authors introduce the notion of topological entropy for NDS (analogy to the definition for ADS, given in [1]) and they show that it is, similarly to the autonomous case, equivalent to the definition using (n, ε)-separated sets (for autonomous case, see [5]). They also prove many properties of topological entropy of NDS, mainly uniformly convergent and equicontinuous ones. Among all the properties proven in [16], let us state at least the relationship between topological entropy of NDS and topological entropy of ADS formed by its uniform limit: $h(f_{1,\infty}) \leq h(f)$.

Thanks to the previously mentioned results and a result by Štefánková, [24], who proved that a uniformly convergent NDS inherits Li-Yorke chaos from its limit system, we know that in NDS positive topological entropy implies Li-Yorke chaos as well as in ADS. To disprove the opposite direction of the implication we use the same counterexample, [22], as in the ADS case as ADS are just a special case of NDS.

To complete the analogy between the ADS and NDS it was left to investigate whether positive topological sequence entropy is equivalent to Li-Yorke chaos. In [23] one implication of this equivalence was disproved and [i] disproves the other one.

3.2 Outline of the construction

In this section, we will briefly sketch the construction of the counterexample. For details, see [i].

First, we will construct the limit map f of the NDS using the blowing-up orbits technique and then we will construct the NDS itself by perturbing f.

Theorem 1. *There is a surjective map f from $C(I)$ such that:*

1. *f has exactly one infinite ω-limit set $\tilde{\omega}$ and it has zero topological entropy;*
2. *f is not LYC;*
3. *the simple set $M(\tilde{\omega})$ generated by the associated system $\mathcal{J}(\tilde{\omega})$ has non-empty interior;*
4. *for the system $\{G_n\}_{n\in\mathbb{Z}}$ of non-degenerate components of $M(\tilde{\omega})$ it holds: $\forall n \in \mathbb{Z}\ f(G_n) = G_{n+1}$.*

Before we delve into the idea of construction, let us note that it is not our result (i.e. result of [i]). Map with these properties is known from e.g. [6].

Sketch of the construction. As outlined in the previous chapter we will construct f as a topological extension of some base map. This base map will be the adding machine α or, more precisely, we get the base map if we extend the adding machine to the middle third Cantor set $Q \subset I$ and linearly extend it to the whole I.

In this setting we can identify (or code) points of Q with sequences of Σ. We take the set of points whose codes end with $\underline{0}$ and denote it by B_0 and similarly we define B_1 as a set of points with codes ending with $\underline{1}$. The union $B := B_0 \cup B_1$ is the set whose full orbit we will blow up.

This full orbit, say $\overline{B}$, is countable, so we can index it with positive integers $\overline{B} = \{b_i\}_{i=1}^{\infty}$. We take a system $\{G_i\}_{i=1}^{\infty}$ of pairwise disjoint compact intervals from $(0,1)$, indexed in such manner that G_i is to the left of G_j if and only if $b_i < b_j$. Then, we can define a map h from Q to $2^{(0,1)}$ such that $h(b_i) = G_i$ and $h(x)$ is a singleton if $x \notin \overline{B}$.

Let us denote $\bigcup h(Q) =: R$. Then, we can interpret the map h^{-1} as a non-decreasing point-valued map from R to Q which is constant on the intervals G_i. We define f' as a map semiconjugate to the adding machine (on Q) by h^{-1} and finally we extend f' continuously and linearly to the whole I, so that it is constant on $[0, \min R]$ and $[\max R, 1]$. This map is the desired f. □

Now, let us proceed to the counterexample.

Theorem 2. *Let f be a surjective continuous map from I to itself, satisfying conditions 1.-4. from Theorem 1. Then there is a nonautonomous system $f_{1,\infty}$ of surjective continuous maps from I to itself, such that*

1. *it converges uniformly to f,*

2. there is a sequence S of positive integers such that $h_S(f_{1,\infty}) \geq \log(3)$,

3. $(I, f_{1,\infty})$ is not LYC.

As with Theorem 1, we will only outline the construction without proving the properties. For the details and proof of properties, see [i].

Sketch of the construction. Let G_j be the sets defined in the proof of Theorem 1, let $|G|$ denote the length of the interval G and put $\varepsilon_0 := |G_0|/3$. Without loss of generality we can assume that G_0 is located to the left of any other G_j (so its code is $\underline{0}$). Since the intervals G_j are pairwise disjoint and $f(G_j) = G_{j+1}$, we may assume that $f|_{G_j}$ is a linear and strictly monotone map onto G_{j+1}.

The construction of the nonautonomous system is conducted by blocks. We start with an increasing sequence $K_0^1 \subset K_0^2 \subset K_0^3 \ldots \subset G_0$ of compact intervals. The center of every K_0^j is the center c_0 of the interval G_0, the length $|K_1^0| > \varepsilon_0$ and $\lim_{n\to\infty} K_0^n = G_0$. For any $j \in \mathbb{Z}$ and any $n \in \mathbb{N}$, let $K_j^n := f^j(K_0^n)$. Because f is a linear map from G_j onto G_{j+1} we have again for any $j \in \mathbb{Z}$ and any $n \in \mathbb{N}$

$$K_j^1 \subset K_j^2 \subset K_j^3 \subset \ldots \subset G_j, \lim_{n\to\infty} K_n^j = G_j$$
$$\text{and } f\colon \ldots \mapsto K_{-2}^n \mapsto K_{-1}^n \mapsto K_0^n \mapsto K_1^n \mapsto \ldots$$

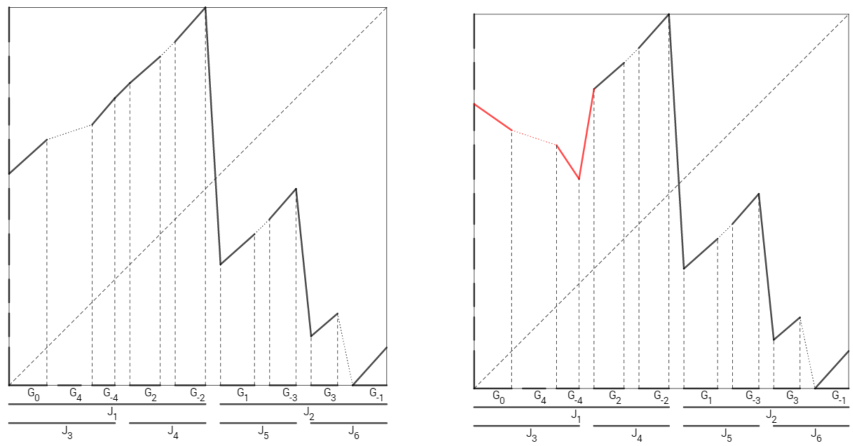
Thanks to the property 1 (Thm. 1) of f there are compact periodic intervals $\{J_i\}_{i\geq 1} \subset \mathcal{I}(\tilde{\omega})$ with codes n_i such that the period of J_i is 2^{k_i} and $\lim_{i\to\infty} |J_i| = 0$.

Let us now define several mappings. First, we define $\tau_{n_k}\colon \Sigma \to \Sigma$ for a block $n_k \in \{0,1\}^k$ as the identity if the first k symbols of a sequence $\underline{n} \in \Sigma$ are distinct from the block n_k and a 0-1-after-k-symbols-reversing map otherwise. To make things clearer, the map τ_{n_k} does not change the first k symbols of a $n_k\underline{n}$ and swaps the zeros to ones and conversely ones to zeros at the very same time in the rest of the sequence, e.g.

$$\tau_{n_k}(n_k110100\ldots) = n_k001011\ldots$$

Next, we define λ_{n_i} using τ_{n_i}, but on I (let us note here that we switched between indeces i and k on purpose, because k refers to the length of the block n_k, while i refers to the index of the corresponding J_i). On the interval J_i it maps the underlying interval G_j (linearly) onto another G_l if and only if τ_{n_i} maps the respective codes one to another as well and it is linearly extended to the rest of the J_i. On the other intervals from the periodic orbit of J_i it is defined as identity and then linearly extended to the whole I. Then, let $\eta_{n_i} := f \circ \lambda_{n_i}$. For the effects of compounding f and λ, see Figure 3.1.

As a next step, we define the maps $\varphi_{i,n}$. Let $\varphi_{i,n}(x) = f(x)$ for $x \notin \text{int}(K_{p_i}^n)$ and let it be a three lap piecewise monotone map on the interior. In the case of $n = 0$ we impose no further restrictions, in all the other cases it maps each of the three parts of $K_{p_i}^n$ obtained by dividing it by $K_{p_i}^{n-1}$ back onto the whole $K_{p_i}^n$, where p_i is an evaluation of n_i (i.e. a sum of the elements of the block



(a) The map f *(b) The map η_{00}*

Fig. 3.1: Sketches of limit map f and its perturbation, dotted lines are unspecified parts of the graph

n_i multiplied by 2 to the power of the position of the element minus 1). With these maps we can define the following sequence $g_{1,\infty}$ for some i and n:

$$g_{1,\infty} = \varphi_{i,n} \circ \lambda_i, \underbrace{\eta_i, \eta_i, \dots, \eta_i}_{2^{k_i}-1 \text{ times}}, \varphi_{i,n} \circ \lambda_i, \underbrace{\eta_i, \eta_i, \dots, \eta_i}_{2^{k_i}-1 \text{ times}}, \varphi_{i,n} \circ \lambda_i, \underbrace{\eta_i, \eta_i, \dots, \eta_i}_{2^{k_i}-1 \text{ times}}, \dots$$

Let $\psi_{i,n} = f$ outside of the interior of $K_{p_i}^{n+1}$, let $\psi_{i,n}|_{K_{p_i}^n}$ be constant with value c_{p_i+1} (the center of G_{p_i}) and let us extend it continuously and linearly on $K_{p_i}^{n+1} \setminus K_{p_i}^n$. The role of $\psi_{i,n}$ is to "kill" LY-scrambled sets. With $\psi_{i,n}$ we finally define the desired NDS. Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers and let $b_n := a_n \cdot 2^k_n + 1$. Then we define $f_{1,\infty}$ as a sequence of blocks B_n of length b_n where:

$$B_n := \varphi_{n,n} \circ \lambda_n, \underbrace{\eta_n, \eta_n, \dots, \eta_n}_{2^{k_n}-1 \text{ times}}, \dots, \varphi_{n,n} \circ \lambda_n, \underbrace{\eta_n, \eta_n, \dots, \eta_n}_{2^{k_n}-1 \text{ times}}, \psi_{n,n}.$$

□

4. NON-INVERTIBLE MINIMAL MAP OF THE KLEIN BOTTLE

In this chapter we briefly review the content of the paper Construction of minimal non-invertible skew-product maps on 2- manifolds, [ii], and also add some related unpublished partial results.

4.1 Historical background

In this section, we will discuss the motivation for [ii] and the partial results and recall its historical background.

The notion of minimality (recalled in chapter Preliminaries) has a clear importance for the discrete dynamics. Hence, during the last decades, much progress has been made in studying minimal subsystems of (M, f) in the case where M is a low dimensional compact connected manifold.

In particular, Auslander and Katznelson have proved [2] that the minimality of (M, f) together with $\dim M = 1$ implies that $M = \mathbb{T}^1$ and that f is conjugate to an irrational rotation (hence, f is a homeomorphism). If $\dim M = 2$ then, due to the Blokh-Oversteegen-Tymchatyn theorem [4], the minimal manifold M must be either the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$. It was also shown in [17] that, in contrast with the minimal system $(\mathbb{T}^1, f)$, there exist minimal fiber-preserving systems $(\mathbb{T}^2, f)$ which are not invertible. The key dynamical and topological components of the proof in [17] are, respectively, the Rees example [20] of a non-distal but point-distal torus homeomorphism and the Roberts-Steenrod characterisation [21] of the monotone transformations of 2-dimensional manifolds.

Since the available constructions [11, 19] of the minimal homemorphisms of the Klein bottle are technically quite involved, the similar question about the existence of minimal non-invertible self-maps of $\mathbb{K}^2$ has been left open in [4, 17]. The main result of paper [ii] is answer to this question proving that there exists a fiber-preserving transformation $\widetilde{S}$ of the Klein bottle, which is minimal and non-invertible.

The results of [17] and [ii] together with the facts that any real analytic montone surjective map on compact connected n-dimensional manifold is a homeomorphism (see [9]) and that a minimal map of 2-dimensional manifold is monotone (see [4]) give rise to a question whether it is possible to have a non-invertible minimal map of 2-dimensional manifold which is differentiable or even of class C^n. Some partial results on this topic are discussed in the third section of this chapter.

4.2 Outline of the construction

In this section, we will briefly sketch the construction of the minimal non-invertible map of the Klein bottle. For details, please see [ii].

In our construction, we will use minimal homeomorphism S of torus (which is factorizable to the Klein bottle) by Parry, [19], and modify it to a non-invertible map $\widehat{S}$ using the adapted measure-theoretical blow-up technique introduced by Hric and Jäger, [14]. This way, we obtain non-invertible minimal map $\widetilde{S}$ of the torus which we finally factorize to the Klein bottle.

Theorem 3. *There exists a minimal non-invertible transformation $\widetilde{S}$ of the Klein bottle $\mathbb{K}^2$.*

Sketch of the construction. First, we will recall the construction by Parry, [19]. Let us take a transformation S of $\mathbb{T}^2$ defined as $S(x,y) = (R(x), \sigma_x(y)) = (x+\alpha, y+r(x))$, where R is a rotation by an irrational angle α and r is mapping from $\mathbb{T}^1$ to $(-1/4, 1/4) \subset \mathbb{R}$, satisfying $r(x) = -r(x+1/2)$. This condition on r ensures commutativity of S with P (see chapter Preliminaries) and hence the factorizability of S to the Klein bottle. Moreover, we require that r's Fourier coefficients satisfy several conditions listed in [19] and lastly we assume that $r(0) = r(1/2) = 0$. In the following we will use notation

$$\sigma_x^n = \sigma_{R^{n-1}(x)} \circ \sigma_{R^{n-2}(x)} \circ \ldots \sigma_{R(x)} \circ \sigma_x.$$

Now, let us take $x_1^* \in (0.1, 0.2) \cap \mathbb{Q}$ and $x_2^* = x_1^* + 1/2$. Next, choose the points $z_1^* = (x_1^*, y_1^*)$ and $z_2^* = P(z_1^*) = (x_2^*, y_2^*)$ such that $y_j^* \neq \sigma^{-m}_{R^m(x_j^*)}(0)$ and $y_j^* \neq \sigma^{-m}_{R^m(x_j^*)}(-r(R^m(x_j^*)))$ for every $m \in \mathbb{Z}$ and $j = 1, 2$. This ensures, that $\mathrm{Orb}_S(z_j^*)$ do not intersect curves $\mathbb{T}^1 \times \{0\}$ and $\{(x, -r(x)) | x \in \mathbb{T}^1\}$ what is important to ensure uniform continuity of $\widehat{S}$.

We will use these points to define curves φ and ψ on the torus. Let φ and ψ have their graphs P-invariant and intersecting in points z_1^* and z_2^*. Moreover, we choose them such that they are equal to zero (or one, which is identified with zero, respectively) except some neighborhood $(\bar{x}_1, \bar{x}_2)$ of x_1^* and $(\bar{x}_1 + 1/2, \bar{x}_2 + 1/2)$ of x_2^* respectively, see Figure 4.1. The last assumption is actually not necessary, but makes things easier to understand.

Now, we define a fibre measure μ_x^0 for the σ-algebra of Borelian subsets of $\mathbb{T}^1$ in the following way on the fibres, where ϕ and ψ are non-zero:

$$\mu_x^0 := \begin{cases} \delta_{y_1^*}, & x = x_1^* \\ \delta_{y_2^*}, & x = x_2^* \\ \frac{\lambda|_{[\psi(x), \phi(x)]}}{\phi(x) - \psi(x)}, & \phi(x) > \psi(x) \\ \frac{\lambda|_{[\phi(x), \psi(x)]}}{\psi(x) - \phi(x)}, & \psi(x) > \phi(x), \end{cases}$$

where δ denotes Dirac measure and $\lambda|_{[a,b]}$ denotes a Lebesgue measure of intersection of the measured set and interval (arc) $[a, b]$ and the inequality is taken

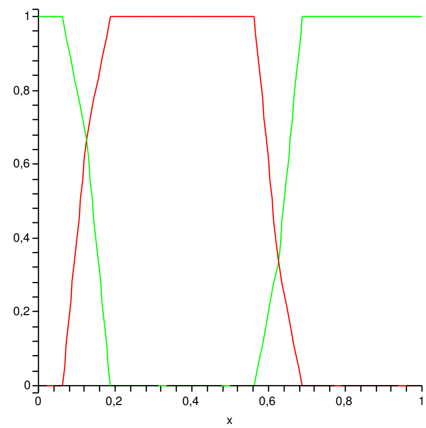


Fig. 4.1: The curves φ and ψ

with respect to the ordering on the interval [0,1). On the rest of the fibres let μ_x^0 be simply Lebesgue measure denoted by λ.

Subsequently, define measures μ_x^n for an integer n as follows:

$$\mu_x^n := \mu_{R^n(x)}^0 \circ \sigma_x^n.$$

And finally, we define a measure μ_x as a weighted sum of the measures μ_x^n and the Lebesgue measure:

$$\mu_x := b\lambda + \sum_{n=0}^{\infty} a_n \mu_x^n,$$

where $a_n \in (0, +\infty)$ are such that $a = \sum_{n \geq 0} a_n < 1$ and $b = 1 - a$.

Following [14], we consider continuous selfmap $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $T(x, y) := (x, \tau_x(y))$, where

$$\tau_x(y) := \min\{y' \in [0,1] | \mu_x[0, y'] \geq y\}.$$

For purposes of this abstract we will only state that the minimum exists and omit the details, which can be found in [ii].

We are ready to construct the non-invertible minimal map $\widehat{S} : \mathbb{T}^2 \to \mathbb{T}^2$. Defining $\widehat{S}$ on the set Λ by

$$\widehat{S}|_\Lambda := T^{-1}|_\Lambda \circ S|_\Lambda \circ T|_\Lambda,$$

we will extend it continuously on the whole torus. This extending is possible thanks to the uniform continuity of $\widehat{S}|_\Lambda$.

So far we have constructed minimal non-invertible map $\widehat{S}$ of $\mathbb{T}^2$. The last step is to factorize the map to the Klein bottle by the usual factorization of torus to $\mathbb{K}^2$ recalled in the chapter Preliminaries. That completes the construction. $\square$

4.3 Unpublished related results

This section summarizes partial results obtained in unfinished later research on the compatibility of smoothnes, minimality and non-invertiblity of maps on 2-dimensional manifolds. Unlike previous parts of this abstract we delve to full details as we cannot refer to any publications.

The furthest result we were able to prove before we discontinued our work for several reasons is the following:

Theorem 4. *The map $\widehat{S}$ constructed in [ii] is not continuously differentiable.*

To prove this statement we will need the following lemma:

Lemma 1. *Let*
$$d_n^* := \mu_{R^{-n}(x_1^*)}[0, y_1^* - r(R^{-n}(x_1^*)))$$
and
$$e_n^* := \mu_{R^{-n}(x_1^*)}[0, y_1^* - r(R^{-n}(x_1^*))].$$
Then, the map $s_n\colon [d_n^*, e_n^*] \to [\mu_{R^{-n+1}(x_1^*)}[0, y_1^*), \mu_{R^{-n+1}(x_1^*)}[0, y_1^*]]$ defined by
$$s_n(y) = proj_2(\widehat{S}(R^{-n}(x_1^*), y)),$$
where $proj_2$ is the second projection map, is linear with slope 2.

Proof. For simplicity, we will prove the lemma only for $n = 1$ and hence we will omit the subscripts n (e.g.: $e^* = e_1^*$). The proof can be generalized, but it gets more technically involved.

It holds $e^* = d^* + 1/8$, because the only difference between these two points is the Dirac measure, which is in this case multiplied by a factor 2^{-2-1}. Images of these points by s are:

$$s(d^*) = \mathrm{proj}_2(\widehat{S}(R^{-1}(x_1^*), d^*)) = \mathrm{proj}_2\left(\lim_{\varepsilon\to 0^+} \widehat{S}(R^{-1}(x_1^* - \varepsilon), d^* - \varepsilon)\right)$$
$$= \mathrm{proj}_2\left(\lim_{\varepsilon\to 0^+} T^{-1}\left(S\left(R^{-1}(x_1^* - \varepsilon), \min\{y' | \mu_{R^{-1}(x_1^*-\varepsilon)}[0, y'] \geq d^* - \varepsilon\}\right)\right)\right)$$
$$= \mathrm{proj}_2\left(\lim_{\varepsilon\to 0^+} T^{-1}(x_1^* - \varepsilon, y_1^* - \delta(\varepsilon))\right) = \mu_{x_1^*}[0, y_1^*).$$

Note that here the δ does not mean Dirac measure, but some positive number as usual in ε-δ criterions. Then, in a completely analogical way we get
$$s(e^*) = \mu_{x_1^*}[0, y_1^*].$$

Before we will continue, let us do two minor calcultions. First, let us compute

$$\mu^0_{x_1^*-\varepsilon}[r(R^{-1}(x_1^* - \varepsilon)), \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon)]$$
$$= \frac{\beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - \psi(x_1^* - \varepsilon)}{\varphi(x_1^* - \varepsilon) - \psi(x_1^* - \varepsilon)} = 1 - \beta,$$

where $\beta \in [0,1]$ and ε is small enough to $r(R^{-1}(x_1^* - \varepsilon)) \leq \psi(x_1^* - \varepsilon)$.

Second, we will show that

$$\lim_{\varepsilon \to 0^+} \min\{y' | \mu_{R^{-1}(x_1^*-\varepsilon)}[0, y'] \geq y\} \stackrel{?}{=} \lim_{\varepsilon \to 0^+} \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon)),$$

where $\beta = (e^* - y)/(e^* - d^*)$ which lies in $[0,1]$ as $y \in [d^*, e^*]$.

But indeed, if we compute the limit of the measure of the expression on the right hand side we get:

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \mu_{R^{-1}(x_1^*-\varepsilon)}[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))] \\
&= \lim_{\varepsilon \to 0^+} \left(\mu_{R^{-1}(x_1^*-\varepsilon)}[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))]\right) \\
&\qquad + \frac{1}{8}\mu^0_{x_1^*}[r(R^{-1}(x_1^*)), y_1^*] - \frac{1}{8}\underbrace{\mu^0_{x_1^*}[r(R^{-1}(x_1^*)), y_1^*]}_{=1} \\
&= \lim_{\varepsilon \to 0^+} \Bigg(\frac{1}{4}\lambda[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))] \\
&\quad + \frac{1}{4}\mu^0_{R^{-1}(x_1^*-\varepsilon)}[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))] \\
&\quad + \frac{1}{8}\mu^0_{x_1^*-\varepsilon}[r(R^{-1}(x_1^* - \varepsilon), \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon)] + \dots\Bigg) \\
&\qquad + \frac{1}{8}\mu^0_{x_1^*}[r(R^{-1}(x_1^*)), y_1^*] - \frac{1}{8} \\
&= \lim_{\varepsilon \to 0^+} \Bigg(\frac{1}{4}\lambda[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))] \\
&\quad + \frac{1}{4}\mu^0_{R^{-1}(x_1^*-\varepsilon)}[0, \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon) - r(R^{-1}(x_1^* - \varepsilon))] \\
&\qquad + \frac{1}{8}\mu^0_{x_1^*}[r(R^{-1}(x_1^*)), y_1^*] + \dots\Bigg) \\
&+ \lim_{\varepsilon \to 0^+} \left(\frac{1}{8}\underbrace{\mu^0_{x_1^*-\varepsilon}[r(R^{-1}(x_1^* - \varepsilon)), \beta\psi(x_1^* - \varepsilon) + (1-\beta)\varphi(x_1^* - \varepsilon)]}_{=1-\beta}\right) - \frac{1}{8} \\
&= \underbrace{\mu_{R^{-1}(x_1^*)}[0, y_1^* - r(R^{-1}(x_1^*)]}_{=e^*} + \frac{1}{8}(1-\beta) - \frac{1}{8} \\
&= \underbrace{e^* - \frac{1}{8}}_{d^*} + \frac{1}{8}\left(1 - \frac{y - d^*}{e^* - d^*}\right) = d^* + \frac{1}{8}\left(1 - \frac{y - d^*}{\frac{1}{8}}\right) = y
\end{aligned}$$

and if we take the same limit of something smaller than the right hand side expression with the same tricks we get:

$$\lim_{\varepsilon\to 0^+} \mu_{R^{-1}(x_1^*-\varepsilon)}[0, \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon) - r(R^{-1}(x_1^*-\varepsilon)) - \delta]$$
$$= d^* - 0 - \frac{1}{8}(1-\beta) - \frac{\delta}{8(e^*-d^*)} = y - \frac{\delta}{8(e^*-d^*)}$$

Now we can continue

$$\begin{aligned}
s(y) &= \mathrm{proj}_2\left(\widehat{S}\left(R^{-1}(x_1^*), y\right)\right) = \mathrm{proj}_2\left(\lim_{\varepsilon\to 0^+} \widehat{S}\left(R^{-1}(x_1^*-\varepsilon), y\right)\right)\\
&= \mathrm{proj}_2\left(\lim_{\varepsilon\to 0^+}(T^{-1}\circ S)\left(R^{-1}(x_1^*-\varepsilon), \min\{y' | \mu_{R^{-1}(x_1^*-\varepsilon)}[0,y'] \geq y\}\right)\right)\\
&= \lim_{\varepsilon\to 0^+} \mu_{x_1^*-\varepsilon}\left[r(R^{-1}(x_1^*-\varepsilon)), \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon)\right]\\
&= \lim_{\varepsilon\to 0^+}\left(\frac{1}{4}\lambda\left[r(R^{-1}(x_1^*-\varepsilon)), \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon)\right]\right.\\
&\quad \left. + \frac{1}{4}\mu^0_{x_1^*-\varepsilon}\left[r(R^{-1}(x_1^*-\varepsilon)), \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon)\right] + \dots\right)\\
&\quad + \frac{1}{4}\mu^0_{x_1^*}[0, y_1^*] - \frac{1}{4}\underbrace{\mu^0_{x_1^*}[0, y_1^*]}_{=1}\\
&= \lim_{\varepsilon\to 0^+}\left(\frac{1}{4}\lambda\left[r(R^{-1}(x_1^*-\varepsilon)), \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon)\right] + \frac{1}{4}\mu^0_{x_1^*}[0, y_1^*] + \dots\right)\\
&\quad + \lim_{\varepsilon\to 0^+}\left(\frac{1}{4}\underbrace{\mu^0_{x_1^*-\varepsilon}\left[r(R^{-1}(x_1^*-\varepsilon)), \beta\psi(x_1^*-\varepsilon) + (1-\beta)\varphi(x_1^*-\varepsilon)\right]}_{=1-\beta}\right) - \frac{1}{4}\\
&= \mu_{x_1^*}[0, y_1^*] - \frac{1}{4} + \frac{1}{4}(1-\beta) = \mu_{x_1^*}[0, y_1^*) + \frac{1}{4}\frac{y-d^*}{e^*-d^*}\\
&= 2y - 2d^* + \mu_{x_1^*}[0, y_1^*).
\end{aligned}$$

□

Now, we are ready to prove Theorem 4.

Proof of the Theorem 4. Due to Lemma 1 we know that the derivative of $\widehat{S}$ in the direction of vertical fibres in the fibres $R^{-n}(x_1^*)$ and arc $[d_n^*, e_n^*]$ for $n \geq 1$ is 2. Whereas the derivative of $\widehat{S}$ in the direction of vertical fibres in the fibre x_1^* in certain arc is 0 (as there is constant part).

Because $\widehat{S}$ is minimal, the backward orbit of every point is dense in $\mathbb{T}^2$. As a result, in every neighborhood of the point in the arc in the fibre x_1^* where the directional derivative is 0 is a point from the arc $[d_n^*, e_n^*]$ in the fibre $R^{-n}(x_1*)$ where the directional derivative is 2. □

Altough the map $\widehat{S}$ is not continuously differentiable we can say it is Lipschitz in the vertical fibres.

Theorem 5. *For any $x \in \mathbb{T}^1$, the map $s\colon \mathbb{T}^1 \to \mathbb{T}^1$ defined as*

$$s(y) = proj_2(\widehat{S}(x,y))$$

is Lipschitz with Lipschitz constant $L = 2$.

Proof. First, let us show useful relation between fibre measures.

$$\begin{aligned}\mu_x[y_1,y_2] &= \frac{1}{2}\lambda[y_1,y_2] + \frac{1}{4}\mu_x^0[y_1,y_2] + \frac{1}{8}\mu_{R(x)}^0[y_1+r(x), y_2+r(x)] + \dots \\ &= \frac{1}{4}\lambda[y_1,y_2] + \frac{1}{4}\mu_x^0[y_1,y_2] + \frac{1}{2}\mu_{R(x)}[y_1+r(x), y_2+r(x)]\end{aligned}$$

One can of course take open or half-open intervals instead of closed ones, but except for the fibres $R^{-n}(x_1^*)$ there is no difference as there are no atoms. We will also use a shortcut $\widehat{\sigma}_x(y) = \mathrm{proj}_2(\widehat{S}(x,y))$.

Now, we can proceed to the proof of Lipschitz continuity of s. Let us take y_1 and y_2 with distance larger than $1/4$ (and smaller than $1/2$ as that is the largest distance possible on $\mathbb{T}^1$). Then obviously, one can take 2 as Lipschitz constant as (again) $1/2$ is the maximal distance of the images.

Next, let us restrain to the fibres $x \neq R^{-n}(x_j^*)$ for $j \in \{1,2\}$ and $n \in \mathbb{N}_0$, and y_1 and y_2 such that $|y_1 - y_2| < 1/4$ and $y_1 < y_2$. Then, we can represent these points as preimages by τ_x of some other points Y_1 and Y_2 respectively. We can do this as τ_x is a bijection for the chosen fibres. Then,

$$y_1 = \mu_x[0, Y_1] \text{ and } y_2 = \mu_x[0, Y_2].$$

Now, we are ready to start our calculation. Let it divide in two cases – the first one when both images are "on the same side of 0" and the second one when $\widehat{\sigma}_x(y_2)$ passes 0, while $\widehat{\sigma}_x(y_1)$ doesn't.

$$\begin{aligned}&|\widehat{\sigma}_x(y_1) - \widehat{\sigma}_x(y_2)| = |\mu_{R(x)}[0, Y_1 + r(x)] - \mu_{R(x)}[0, Y_2 + r(x)]| \\ &\quad = \mu_{R(x)}[Y_1 + r(x), Y_2 + r(x)] = 2\mu_x[Y_1,Y_2] - \frac{1}{2}\mu_x^0[Y_1,Y_2] - \frac{1}{2}\lambda[Y_1,Y_2] \\ &\quad\quad \leq 2\mu_x[Y_1,Y_2] = 2|\mu_x[0,Y_1] - \mu_x[0,Y_2]| = 2|y_1 - y_2|\end{aligned}$$

$$\begin{aligned}&1 - |\widehat{\sigma}_x(y_1) - \widehat{\sigma}_x(y_2)| = 1 - |\mu_{R(x)}[0, Y_1 + r(x)] - \mu_{R(x)}[0, Y_2 + r(x)]| \\ &\quad = 1 - \mu_{R(x)}[Y_2 + r(x), Y_1 + r(x)] = \mu_{R(x)}[Y_1 + r(x), Y_2 + r(x)] \\ &\quad = 2\mu_x[Y_1,Y_2] - \frac{1}{2}\mu_x^0[Y_1,Y_2] - \frac{1}{2}\lambda[Y_1,Y_2] \leq 2\mu_x[Y_1,Y_2] \\ &\quad\quad = 2|\mu_x[0,Y_1] - \mu_x[0,Y_2]| = 2|y_1 - y_2|\end{aligned}$$

Let us now consider case where $1 - |y_1 - y_2| < 1/4$. It is again divided into to parts as the previous case.

$$\begin{aligned}&1 - |\widehat{\sigma}_x(y_1) - \widehat{\sigma}_x(y_2)| = 1 - |\mu_{R(x)}[0, Y_1 + r(x)] - \mu_{R(x)}[0, Y_2 + r(x)]| \\ &\quad = 1 - \mu[Y_1 + r(x), Y_2 + r(x)] = 1 - 2\mu_x[Y_1,Y_2] + \frac{1}{2}\mu_x^0[Y_1,Y_2] + \frac{1}{2}\lambda[Y_1,Y_2] \\ &\quad\quad \leq 2 - 2\mu_x[Y_1,Y_2] = 2(1 - \mu_x[Y_1,Y_2]) = 2(1 - |y_1 - y_2|)\end{aligned}$$

$$\begin{aligned}|\widehat{\sigma}_x(y_1) - \widehat{\sigma}_x(y_2)| &= |\mu_{R(x)}[0, Y_1 + r(x)] - \mu_{R(x)}[0, Y_2 + r(x)]| \\ &= \mu_{R(x)}[Y_2 + r(x), Y_1 + r(x)] = 1 - \mu_{R(x)}[Y_1 + r(x), Y_2 + r(x)] \\ &= 1 - 2\mu_x[Y_1, Y_2] + \frac{1}{2}\mu_x^0[Y_1, Y_2] + \frac{1}{2}\lambda[Y_1, Y_2] \\ &\leq 2 - 2\mu_x[Y_1, Y_2] = 2(1 - \mu_x[Y_1, Y_2]) = 2(1 - |y_1 - y_2|)\end{aligned}$$

These calculations may seem tricky if we realize that we y_1 and y_2 don't lie in $\mathbb{R}$, but in $\mathbb{T}^1$ instead. But the distances are real numbers and due to our assumptions they all lie in the interval [0,1/2], so there is no sense (and thus no problem) taking the mod1 operation. The only place where the mod1 has sense is the sum $Y_i + r(x)$ which we took into account.

We are finally left with the other fibres, i.e. $x = R^{-n}(x_j^*)$ for $j \in \{1, 2\}$ and $n \in \mathbb{N}_0$. We can use the same computation on them except for arcs of length $1/2^{n+2}$, where the y's can't be represented as a measure due to its non-surjectivity. But on these arcs is the map $\widehat{\sigma}_x$ either linear with slope 2 due to Lemma 1, or constant (the fibre x_1^*). □

So far, this is all we currently know about the map $\widehat{S}$ and the possibility of having a differentiable, non-invertible minimal map of $\mathbb{T}^2$ or $\mathbb{K}^2$.

To conclude this section, chapter and the whole abstract, we would like to sum up possible following research. The cornerstone could be the papers by Church on differentiable monotone maps [8, 9, 10] and interchanging analyticity for continuous differentiability in the proofs or finding the obstacles to do so.

PUBLICATIONS CONSTITUTING THE BODY OF THE THESIS

Publications constituting the body of the thesis and my percentage contribution towards each of them are listed below.

[i] J. Šotola, Relationship between Li-Yorke chaos and positive topological sequence entropy in nonautonomous dynamical systems, Disc. Cont. Dyn. Sys 38 (2018), to appear, 100 %

[ii] J. Šotola, S. Trofimchuk, Construction of minimal non-invertible skew-product maps on 2- manifolds, Proc. Amer. Math. Soc. 144 (2016), 723–732, 67 %

PRESENTATIONS RELATED TO THE THESIS

- 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, 7th–11th July 2014, Madrid, Spain,
  Talk: On the Minimal Non-invertible Skew-products of 2-manifolds
- 18th Czech-Slovak Workshop on Discrete Dynamical Systems, 8th–12th September 2014, Malenovice, Czechia,
  Talk: On the Minimal Non-invertible Skew-products of 2-manifolds
- 6th Visegrad Conference on Dynamical Systems, 6th–10th July 2015, Praha, Czechia,
  Talk: On the minimal non-invertible skew-products of 2-manifolds
- 21st International Conference on Difference Equations and Applications, 19th–25th July 2015, Bialystok, Poland,
  Talk: On the minimal non-invertible skew-products of 2-manifolds
- 19th Czech-Slovak Workshop on Discrete Dynamical Systems, 12th–16th September 2016, Karlova Studánka, Czechia,
  Talk: On the construction and differentiability of minimal non-invertible skew-product maps of 2-manifolds
- Czech, Slovenian, Austrian, Slovak and Catalan Mathematical Societies joint meeting, 20th–23rd September 2016, Barcelona, Spain,
  Talk: On the construction and differentiability of minimal non-invertible skew-product maps of 2-manifolds

PAPERS CITING THE PUBLICATIONS CONSTITUTING THE BODY OF THE THESIS

- J. Šotola, S. Trofimchuk, Construction of minimal non-invertible skew-product maps on 2- manifolds, Proc. Amer. Math. Soc. 144 (2016), 723–732 is cited by:
  - J. P. Boroński, A. Clark and P. Oprocha, A Compact minimal space Y such that its square $Y \times Y$ is not minimal, Adv. Math., to appear, preprint: arXiv:1612.09179

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