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REPORT

on Ph.D. Thesis of **Jana Hantáková** entitled
"Distributional chaos in compact metric spaces"

Ph.D. Thesis of Jana Hantáková was prepared under supervision of Professor Jaroslav Smítal and is a composition the following 4 research articles:

(A) Jana Doleželová, *Distributionally scrambled invariant sets in a compact metric space.* Nonlinear Anal. **79** (2013), 80–84.
(B) Jana Doleželová, *Scrambled and distributionally scrambled n-tuples.* J. Difference Equ. Appl. **20** (2014),1169–1177.
(C) Jana Doleželová-Hantáková, *Distributional chaos and factors.* J. Difference Equ. Appl. **22** (2016), 99–106.
(D) Jana Hantáková, *Iteration problem for distributional chaos*, unpublished manuscript.

All four articles describe some selected aspects of distributional chaos. Thesis contains also an 8 pages introduction, summarizing main results contained in these papers and explaining motivation and historical background behind considered notions.

Article (A) is devoted to construction of invariant and uncountable DC1-scrambled set under some mild condition. The author shows that if a dynamical system has weak specification property and the set of periods is infinite then there exists a Mycielski DC1-scrambled set which is invariant and dense. This result provides an affirmative answer to an open question stated by this referee in 2009.

It is also worth mentioning that first Remark in Section 5 of the introduction is not completely correct. Namely, Theorem 1 is not consequence of [13]. While the authors of [13] use the name weak specification, they mean Definition 4 with arbitrarily large $s > 1$ but without condition (P), while WSP considered in Theorem 1 uses this condition with $s = 2$. It is not clear how to reproduce "periodic-like" behavior necessary in Theorem 1 with WSP, and [13] does not provide any answer to that question. Additionally authors of [13] assume that f is surjective, which is not the case of Theorem 1 (and it is not completely clear if WSP is hereditary on $\bigcap_{n=0}^{\infty} f^n(X)$). In that sense results of (A) are nice and in to some extent optimal.

In article (B) the author considers a natural generalization of definition of DC1 pair to n-tuples. The main idea is that points in the trajectory of the tuple should be all together ε-close over a set of iterations of upper density one for any $\varepsilon > 0$, and for some $\delta > 0$ there should be a set of iterations of upper density one, where all the trajectories are δ-apart. Strict mathematical definition of this intuition is expressed in Definition 3 (see thesis introduction). As the main result of article (B) the author constructs a special set of points (in full shift), limiting on the Morse minimal system, whose closure provides a subshift with the property that it contains an uncountable DC1-scrambled set but no scrambled triples (in the sense of Li and Yorke). Previously it was known that there are systems with DC1-scrambled set but without DC1-triples. Therefore, article (B) provides a more delicate example by a completely different method of construction. The proof is very long and technical, which numerous intermediate steps of the construction. This clearly proves that the author has very good understanding of dynamics of shift spaces and mechanism behind appearance of chaotic pairs and tuples.

In article (C) the author studies relations between chaotic properties of dynamical systems and their extensions. While it is well known that topological entropy of an extension cannot decrease (so this type of chaos survives), the author shows that similar statement is not true for distributional chaos. She provides in article (C) an example of a dynamical system which is DC1-chaotic but at the same time it has extension without even single DC2-pair. It is obtained by a very interesting application of additional rotation of points in the extension. This ensures sufficiently often separation of trajectories of distinct points, which is in obvious opposition to one of the conditions in the definition of DC2. Again the idea of construction is very interesting and details are quite complicated and technical. Once again it proves high mathematical maturity of the author, her independent thinking and good mathematical skills.

The last article (D) considers a version of distributionally chaotic pair placed between notions of DC2 and DC3 pairs, which is called in the literature DC$2\frac{1}{2}$. While it seems that the author is responsible for appearance of this definition in the literature (with some other co-authors), this first paper on the topic is not the part of the thesis (it is [14] in the bibliography of the introductory part of the thesis). The topic of article (D) is description of some remaining properties of DC$2\frac{1}{2}$ in relation to DC3. Strictly speaking, the author proves in article (D) that DC$2\frac{1}{2}$ pairs are preserved under higher iterations. She also provides an example which has DC3 pair with respect to f^2 but not DC3 pair for f. It is known that converse is impossible, because DC3 pair for f gives raise to DC3 pair for f^2 (however not necessarily initial (x, y) is such a pair). She also proves that infinite DC3-scrambled set for f generates an infinite DC3-scrambled set for f^n for each $n > 1$, however it remains an open question whether the same is true for uncountable DC3-scrambled sets. Again, construction of the above mentioned example in article (D) is highly nontrivial.

Let me state now some more detailed remarks. Since article (D) is the only unpublished manuscript, I will restrict my comments to this article. In what follows, page and line number are presented in respect to that article:

— Abstract should present only results of the article, to provide good overview what was done. General remarks about history of research and theoretical background should be postponed to introduction section.
— p1.14: "started with **paper of** Li and Yorke".
— p1.25: "are well defined" suggests that other notions were not well defined. It seems that the authors want to say here that the notion "behaves well", e.g. is preserved by conjugacy, etc.
— p1.-11: Statement "function f is DC3" uses undefined notion. It should be said here that f **has DC3 pair**.
— p1.-8: Statement "...that DC3 does not imply.." should be rephrased. First of all the sentence refers to DC3 pair, not some abstract DC3 property. Second problem is that standard definitions of chaos (distributional, in the sense of Li and Yorke, etc.) involve uncountable scrambled sets, not single pairs. While some authors define chaotic dynamics by existence of one chaotic pair, it is not historically justified approach.
— p1.-4: classical definition of distribution functions $\Phi(t)$ and $\Phi^*(t)$ is usually stated

(and well defined, also in the thesis) for all parameters $t \in \mathbb{R}$. Range of t is not explicitly specified in Definition 1. This makes all further comments and definitions nonstandard.

— p2.3: "Another strengthened **version** of distributionally **chaotic pair**"
— p2.6: word "cited" should be removed
— p2.7: it is not clear what the author means by "nowadays" since uncountable scrambled sets are requirement of chaos from very beginning.
— Definition 1: in the definition a pair (x_1, x_2) appears, while a few lines above it was (x, y). The notation should be unified.
— p2.-16: "cardinality of **a** set A".
— p2.-8: "where $0 \leq a < b < \text{diam}X$" suggests that the condition is satisfied for all such a, b, while in the definition we certainly want **some** a, b.
— p2.-6: "scrambled pair" is not defined.
— p3.6: "Li-Yorke scrambled" is not defined.
— p3.13: Sentence "Likewise DC1 and..." refers to some result. Was this result proved somewhere? A reference or a proof should appear here.
— Statement of Theorem 1 is weaker than what is actually proved. It follows from the proof that the same pair (x, y) is good for f and f^N, while the statement as it is does not guarantee that.
— p4.4: The proof should be a little more careful, since $\frac{n}{N}$ is not always an integer.
— Lemma 3: It would be helpful to emphasize that $\Phi^*(\delta) > c$ for every $\delta > 0$.
— p4.-2: Should be "$\Psi(\delta) \leq \Psi(q)$".
— p5.5: "There are only **a** few".
— p5.8: "DC3 system" is an undefined notion.
— p5.15: Statement "Dynamical system O_1" is unclear. Dynamical systems is by convention a pair consisting of a space and a map.
— p5.17: Notation $[x, 0]$ is confusing, like in $F([x, 0])$. It suggest an interval, while the author means two coordinates, denoted by standard notation as $(x, 0)$.
— p5.-16: "Other points in O_1 are fixed". Is O_1 a space or a dynamical system?

The above list is by no means complete. It should give the author some guidelines how to improve the presentation in the article before publication. I hope she will find them helpful.

In total, I have enjoyed reading this thesis. It is a nice combination of a few advanced constructions and interesting theorems. In my opinion articles (A)-(D) are more than enough that can be expected from candidate to Ph.D. degree. It should be also stressed out that in all four articles Jana Hantáková (formerly Doleželová) appears as a single author, and additionally all of them are motivated by recent research results by other authors.

In the view of the above I consider that Jana Hantáková deserves to obtain the Ph.D. degree in mathematics. Therefore I recommend to award the Ph.D. degree.