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Contents

1. Introduction

This Thesis is based on three independent papers [1]–[3] (two are published and one is submitted). These papers are connected by one common subject-discrete dynamical systems generated by continuous maps.

The first paper [1] is devoted to so called Sharkovsky's program of classification of triangular maps. We consider two properties which are equivalent to zero topological entropy for interval maps and was not include in the original program. On one hand we show that two implications remain true when we consider triangular map of the square on the other hand, we give two counterexamples which show that the remaining two implications are not true in the case of triangular map of the square.

In the second paper [2] we study periodicity in nonautonomous dynamical systems. We propose a new definition of periodic point in nonautonomous dynamical system - asymptotically periodic point. We show that uniformly convergent nonautonomous dynamical system which is topologically transitive and has dense set of asymptotically periodic points has sensitive dependence on initial condition. Also, we give a counterexample which show that the Sharkovsky's Theorem is not valid in the case of nonautonomous dynamical system for various definitions of periodic point.

The last paper [3] is devoted to transitivity in uniformly convergent nonautonomous discrete dynamical systems. We investigate which condition ensure that the property of topological transitivity is inherited from nonautonomous dynamical systems, respectively from maps which generate the nonautonomous dynamical system to limit map or vice versa.

2. Basic terminology and notation

Throughout the abstract, let (X, d) be a compact metric space with metric d and $I = [0, 1]$ be the compact unit interval. By $C(X)$, resp. by $C(I)$, we denote the class of continuous maps $X \to X$, resp. $I \to I$. The set of all natural number is denoted by $\mathbb{N}$ and the set of all non-negative integers is denoted by $\mathbb{N}_0$. Let $f \in C(X)$ by a *(discrete) autonomous dynamical system*, briefly ADS, we mean the pair (X, f). For any $n \in \mathbb{N}_0$ and $f \in C(X)$ we denote the n-th iteration of $x \in X$ under f by $f^n(x)$. The *trajectory* of $x \in X$ under f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$.

Denote by $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps $f_i \in C(X)$. By a *nonautonomous dynamical system*, briefly NDS, we mean a pair

$(X, f_{1,\infty})$. For $f_{1,\infty}$ and any $i, n \in \mathbb{N}$ we put $f_i^0 = id_X$ and denote by $f_i^n = f_{i+(n-1)} \circ \cdots \circ f_{i+1} \circ f_i$ the n-th iteration of $f_{1,\infty}$ starting from f_i. The *trajectory* of x under $f_{1,\infty}$ is the sequence $\{f_1^n(x)\}_{n=1}^{\infty}$. A autonomous dynamical system is special case of NDS with $f_n = f$ for every $n \in \mathbb{N}$.

Another terminology and notation is given in next sections when needed.

3. Triangular maps

The *skew-product* map is a continuous map $F : X \times Y \to X \times Y$ of the form $F(x, y) = (f(x), g_x(y))$, where the map $f \in C(X)$ is called *base* of F and $g_x : x \times Y \to Y$, $x \in X$ is a family of continuous maps depending continuously on x. Note that any skew-product map $F(f, g_x)$ naturally defines nonautonomous dynamical systems. Namely for any $x \in X$ put $g_{1,\infty} = \{g_{f^n(x)}\}_{n=0}^{\infty}$.

The *triangular map of the square*, briefly triangular map, is a special case of skew-product map with $X = Y = [0, 1] = I$. We assume that the space I^2 is endowed with maximum metric, i.e., $d((x_1, y_1), (x_2, y_2,)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. The class of triangular maps is denoted by $C_{\triangle}(I^2)$.

Let $f \in C(X)$, $\varepsilon > 0$ and $A = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. A set $E \subset X$ is called (A, n, ε, f)-*separated* if for any two distinct points $x, y \in E$ there is a non-negative integer $0 \leq i < n$ such that $d(f^{a_i}(x_1), f^{a_i}(x_2)) > \varepsilon$. Denote by $s(A, n, \varepsilon, f)$ the maximal cardinality of an (A, n, ε, f)-separated set in X. The *topological sequence entropy of f with respect to the sequence A* is defined by

$$h_A(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log s(A, n, \varepsilon, f).$$

If $A = \{i\}_{i=1}^{\infty}$ then $h_A(f) = h(f)$ and $h(f)$ is called *topological entropy of f*. Let $h_{\infty}(f) = \sup_{A \in \mathcal{A}} h_A(f)$, where $\mathcal{A}$ is the set of all strictly increasing sequences of positive integers.

Let $n \geq 2$, an n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ is called *LY-scrambled* if

$$\limsup_{k \to \infty} \min_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) > 0,$$

$$\liminf_{k \to \infty} \max_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) = 0.$$

A system (X, f) is *n-chaotic in the sense of Li-Yorke*, briefly *LY-n-chaotic*, if there is an uncountable set $S \subset X$ such that every essential n-tuple $(x_1, x_2, \ldots, x_n) \in S^n$ (i.e., $x_i \neq x_j$ for any $1 \leq i < j \leq n$) is LY-scrambled.

It is known that, for interval maps f, there are more than 50 properties which are equivalent with zero topological entropy of f. More than 40 of them are applicable to triangular maps. In 1989 A. N. Sharkovsky proposed a program of a classification of the triangular maps of the square with respect to these applicable properties. This program was completed in 2013, for a survey see [16]. However, the Sharkovsky classification program does not include all properties of interval maps which are equivalent to zero topological entropy which are known today. In [1] we study two of them and we give an answer to question whether these properties are valid for triangular maps of the square.

(1) [18] Let $I = [0, 1]$ and $f : I \to I$ be continuous. Then $h(f) = 0$ if and only if there is no LY-scrambled triple for f.
(2) [17] Let $I = [0, 1]$ and $f : I \to I$ be continuous. Then $h(f) = 0$ if and only if $h_\infty(f)$ is finite.

Theorem A.

(i) Let $F \in C(X)$ has no LY-scrambled triple. Then $h(F) = 0$.
(ii) There exists a map $F \in C_\triangle(I^2)$ which has a scrambled triple and $h(F) = 0$.

Theorem B.

(i) Let $F \in C(X)$ with $h_\infty(F) < \infty$. Then $h(F)=0$.
(ii) There exists $F \in C_\triangle(I^2)$ with $h(F)=0$ and $h_\infty(F)$ unbounded.

4. Nonautonomous dynamical systems

We consider a particular case of nonautonomous dynamical systems so called uniformly convergent, i.e., $f_{1,\infty}$ uniformly converges to a continuous map f. Further we assume that f_n is surjective for every $n \in \mathbb{N}$, such assumption is very natural since we want to avoid some trivial nonautonomous systems (e.g. if f_1 is constant map then the whole dynamics of $(X, f_{1,\infty})$ is shrunk to single trajectory).

A nonautonomous dynamical system $(X, f_{1,\infty})$ satisfies condition (CC*) if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $k \in \mathbb{N}$ and every $x \in X$, $d((f_n)^k(x), f^k(x)) < \varepsilon$, where $(f_n)^k$ is k-th iteration of map f_n. A nonautonomous dynamical system $(X, f_{1,\infty})$ is *topological transitive* if for every pair of nonempty open sets $U, V \in X$, there is a positive integer n such that $f_1^n(U) \cap V \neq \emptyset$. We say that a nonautonomous dynamical system $(X, f_{1,\infty})$ has *sensitive dependence on initial conditions*, briefly sensitive, if there is $\delta > 0$ such that for any $x \in X$ and $\varepsilon > 0$ there is $y \in X$ with $d(x, y) < \varepsilon$ such that $d(f_1^n(x), f_1^n(y)) > \delta$ for some $n \geq 0$. A nonautonomous dynamical system $(X, f_{1,\infty})$ is *Devaney chaotic* if it satisfies the following conditions:

(i) $(X, f_{1,\infty})$ is topologically transitive,
(ii) $(X, f_{1,\infty})$ has a dense set of periodic points,
(iii) $(X, f_{1,\infty})$ is sensitive.

For an autonomous dynamical system (X, f) the definition of periodic point $x \in X$ is unique and natural (x is *periodic with period* n if n is the smallest natural number such that $f^n(x) = x$). In the case of nonautonomous dynamical system $(X, f_{1,\infty})$ the situation is more complicated and several notation of periodicity have been introduced (see [19], [20], [21], [22]). In [2] we propose a new definition of periodicity in nonautonomous dynamical system $(X, f_{1,\infty})$, so called asymptotic periodicity. A point $x \in X$ is *asymptotically periodic* with the cycle $x_1, x_2 \dots, x_r$ if there exist $x_1, x_2, \dots, x_r \in X$ such that for any $\varepsilon > 0$ there is $n_0 > 0$ such that $d(f_1^{nr+i}(x), x_{i+1}) < \varepsilon$ for $n \geq n_0$ and $0 \leq i < r$. Denote by $AP(f_{1,\infty})$ the set of asymptotically periodic points of $f_{1,\infty}$.

Banks et al., proved in [15] that for ADS (X, f), where X is compact metric space without isolated points, the third condition in the definition of Devaney chaos is redundant. A natural question is whether the same property is valid for NDS. In [2] we give the following answer.

Theorem C. *Let $f_{1,\infty}$ be a nonautonomous system on a compact metric space X without isolated points. Suppose that f_n converges uniformly to $f \in C(X)$. If*

(1) $f_{1,\infty}$ is topologically transitive and
(2) $AP(f_{1,\infty})$ is dense

then $f_{1,\infty}$ is sensitive.

Remark. For interval ADS it is known (see [23]) that topological transitivity implies a dense set of periodic points. We show that in the case of NDS such property is no longer true if we consider arbitrary definition of periodicity which do not admit dense orbit of periodic point.

One of the most famous and important theorem of the theory of interval autonomous dynamical system is Sharkovsky's Theorem. The following Theorem shows that in the case of interval nonautonomous dynamical system such theorem is not true when asymptotic periodicity is considered.

Theorem D. *There exists a nonautonomous dynamical system $(I, f_{1,\infty})$ which uniformly converges to $f(x) := 1-x$ and with asymptotic 2-cycle but without any asymptotic fixed point.*

It is known that even for interval uniformly convergent NDS $(X, f_{1,\infty})$ the topological transitivity of the system $(X, f_{1,\infty})$ nor the topological transitivity of f_n for any n is not sufficient to ensure the topological transitivity of the limit map f or vice versa. Natural question is if there is what property ensure the inheritance of topological transitivity. Fedeli and Le Donne in [24] proved that if NDS $(X, f_{1,\infty})$ satisfies condition (CC*) and f_n is topologically transitive for every $n \in \mathbb{N}$ then the limit map f is topologically transitive. In [3] we proved that even the converse implication is true for piecewise monotone interval map.

Theorem E. *Let f, $f_n \in C(I)$, $n \in \mathbb{N}$ be surjective maps such that f is uniform limit of $f_{1,\infty}$. If f is transitive, piecewise monotone map and $f_{1,\infty}$ satisfies condition (CC*), then there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ $f_n(x) = f(x)$.*

Remark. Furthermore, we give a counterexample on Cantor set which shows that in general case the converse implication is not true.

5. Publications concerning the Thesis

[1] V. Pravec. *On dynamics of triangular maps of the square with zero topological entropy*, Qualitative Theory of Dynamical Systems **18** (2019), 761–768.

[2] V. Pravec. *Remarks on definitions of periodic points for nonautonomous dynamical system*, Journal of Difference Equation and Applications, **25** (2019), 1372–1381.

Cited in:
X.M. Yuan, X. Zhu, C. Wang, L.J. Zhang. *Research on Dynamic Modeling and Parameter Influence of Adaptive Gun Head Jet System of Fire-Fighting Monitor*, IEEE ACCESS, **8** (2020), 121182–121196.

[3] M. Mlíchová, V. Pravec. *Transitivity in nonautonomous dynamical system* (submitted in January 2021)

6. Other publication

[4] V. Pravec, J. Tesarčík. *On distributional spectrum of piecewise monotonic maps*, (submitted to Ergodic Theory and Dynamical Systems in December 2020)

7. Conferences

[5] 21th European Conference on Iteration Theory, Innsbruck, Austria, September 4–10, 2016.
Talk on: "On dynamics of triangular maps of the square with zero topological entropy."

[6] Czech-Slovak Workshop on Discrete Dynamical Systems, Karlova Studánka, Czech Republic, September 12–16, 2016.
Talk on: "On dynamics of triangular maps of the square with zero topological entropy."

[7] 23rd International conference on different equations and applications, Timmisoara, Romania, July 24–28, 2017.
Talk on: "On dynamics of triangular maps of the square with zero topological entropy."

[8] Wandering Seminar, Lodz, Poland, December 7–10, 2017.

[9] Czech-Slovak Workshop on Discrete Dynamical Systems, Bánska Bystrica, Slovakia, June 19–22, 2018.

[10] 8th Visegrad conference on Dynamical systems, Budapest, Hungary, June 24–28, 2019.
Talk on: " Remark on definitions of periodic points for nonautonomous dynamical system."

[11] Dynamics, Geometry and Analysis: 20 years of Mathematical Institute in Opava, Hradec nad Moravicí, Czech Republic, September 8–13, 2019.
Talk on: " Remark on definitions of periodic points for nonautonomous dynamical system."

[12] Dynamics, Equations, Applications, Kraków, Poland, September 16–20, 2019.
Talk on: " Remark on definitions of periodic points for nonautonomous dynamical system."

[13] 17th International conference on Numerical Analysis and Applied Mathematics, Rhodes, Greece, September 23–28, 2019.

Talk on: " Remark on definitions of periodic points for nonautonomous dynamical system."

8. Long-term visits

[14] Universität Wien, March–May 2017 and October 2018–March 2019, Wien, Austria.

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