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Report on H.V. Le's Habilitation's thesis.

This thesis consists of six papers. This report begins by comments on each separately. Throughout we denote a closed symplectic manifold (M, ω) and its group of symplectomorphisms (i.e. ω-preserving diffeomorphisms) by $\mathrm{Symp}(M, \omega)$. The identity component of $\mathrm{Symp}(M, \omega)$ is called $\mathrm{Symp}_0(M, \omega)$.

1 *Symplectic fixed points, the Calabi invariant and Novikov homology.*

One of the motivating problems in the recent spectacular development of symplectic topology was Arnold's conjecture, which proposed that the number of fixed points of a generic Hamiltonian symplectomorphism ϕ of a closed symplectic manifold (M, ω) should be at least the sum $\sum_i \mathrm{rk}\, H_i(M; \mathbb{Q})$ of the Betti numbers. For this to hold, $\phi \in \mathrm{Symp}_0(M, \omega)$ must be *Hamiltonian*, which means that it is connected to the identity by a path $\phi_t, t \in [0,1]$, with zero flux through all 1-cycles $\gamma : S^1 \to M$. Thus

$$\int_{S^1\times[0,1]} u^*(\omega) = 0, \text{ where } u(s,t) = \phi_t(\gamma(s)).$$

After important work by Floer and many others (including Hofer, Salamon, Ruan, Tian, Fukaya and Ono), this form of Arnold's conjecture was finally established in the mid 1990s. The main breakthrough here was Floer's introduction of what is now called Hamiltonian Floer homology. He established its existence for *spherically monotone* manifolds, i.e. symplectic manifolds in which the first Chern class is a (positive) multiple of the symplectic class $[\omega]$. Subsequent developments all rely on extending the definition of Floer homology to wider classes of manifolds.

Note that the result does not hold for arbitrary elements of $\mathrm{Symp}_0(M, \omega)$. For example, rotations of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ have no fixed points.

This paper (written with Ono) establishes a version of Arnold's conjecture for arbitrary elements of $\mathrm{Symp}_0(M, \omega)$ in the spherically monotone case. The result also has to be appropriately changed: the lower bound for the number if fixed points is no longer the sum of the usually Betti numbers, but is the sum of the Novikov–Betti numbers, i.e. the ranks of the so-called Novikov homology of (M, ω) in the Flux (or Calabi) class $\eta \in H^1(M; \mathbb{R})$ of ϕ.

To prove this the authors must develop a Novikov–Floer theory for symplectomorphisms. The ordinary topological version of Novikov homology was also fairly new at the time, and the appendices to this paper develop some of its properties. At the time, when Hamiltonian Floer theory itself was not well understood, this paper was a very significant advance. I certainly read it very carefully when it came out. Many analytical problems and algebraic problems need to be overcome to show that the homology is well defined; in particular one needs to choose a suitable coefficient ring for the Floer chain groups and also need to establish a corresponding energy identity (Lemma 5.4). It requires a very sure hand and deep understanding to deal with the questions in a consistent manner.

This paper was the first to formulate this Novikov version of Floer theory. This has proved to be a very useful tool in certain situations; in particular Ono recently used it to prove the Flux conjecture.

2. *Cup-length estimate for symplectic fixed points*

The Arnold conjecture mentioned above holds for generic ϕ. In particular, one needs each fixed point of ϕ to be nondegenerate (i.e. the graph of ϕ must be transverse to the diagonal.) If this condition does not hold then, intuitively speaking, certain fixed points can coincide so that there are fewer overall. One form of Arnold's conjecture say that for Hamiltonian $\phi \in \mathrm{Symp}_0(M,\omega)$ the number of fixed point should be at least the cup length of M. (This is the minimum integer m such that $a^m = 0$ for all $a \in H^*(M)$. Its value depends on the coefficients used for $H^*(M)$.)

There is a close analogy here with what happens for functions $f : M \to \mathbb{R}$. If f is nondegenerate (i.e. Morse) then its number $\#\mathrm{Crit}\,(f)$ of critical points is bounded below by the sum of the Betti number of M, while if it is degenerate one can use Lusternick–Schnirelmann theory to show that $\#\mathrm{Crit}\,(f)$ is bounded by the cup length of the cohomology ring.

The question of finding lower bounds in the degenerate case is more tricky than the nondegenerate case. Nevertheless Floer found a way to make it accessible in the case of $\mathbb{Z}/2$ coefficients by establishing the existence of a cap product operation on Floer theory.

This paper (written with Ono) extends his argument to the Novikov case, by showing that such an operation also exists in the theory developed in 1. Thus this paper is in some sense routine. On the other hand there are still many analytic details to check (for example, developing a manageable but sufficiently broad class of perturbations to achieve transversality) that at the time were not in the literature. Hence it is a useful contribution.

3. *Parameterized Gromov–Witten invariants and topology of symplectomorphism groups.*

This paper (written with Ono) develops the very natural idea that Gromov–Witten invariants can be used to probe the topology of $\mathrm{Symp}(M,\omega)$ since they give rise to natural characteristic classes on the classifying space $B\mathrm{Symp}_0(M,\omega)$. The basic idea is that although $B\mathrm{Symp}_0(M,\omega)$ is not a symplectic manifold, the universal M-bundle over it does carry a natural symplectic structure on its fibers, so that one can define a fiberwise version of the Gromov–Witten invariants. These can be used to define cohomology classes on $B\mathrm{Symp}_0(M,\omega)$. Although Le–Ono do not define such classes here,[1] they are implicit in this paper and are used to show that certain elements of $\pi_*(B\mathrm{Symp}_0(M,\omega))$ do not vanish.

In particular, they give a very elegant proof of the nonvanishing of a certain element in $\pi_2(B\mathrm{Symp}_0(S^2 \times S^2,\omega))$ that was first detected by Gromov.

[1] Later on, inspired by a first version of their paper as well as work of Reznikhov and Kedra, I did formulate these Gromov–Witten classes.

In some sense their argument is the same as Gromov's, in that it relies on the same analytic facts. But the framework is much more general, and so applies much more widely.

This paper develops an important link between ideas in topology and the more analytical methods of symplectic topology.

The first version of the paper was very sketchy. This one is much more complete. Again the details of the proof are fairly routine, but they are still not well understood, especially by people who know topology. Hence this paper was well worth rewriting in detail.

4. *Almost complex structures which are compatible with Kähler or symplectic structures.*

This paper (written with Connolly and Ono) discusses the question of which homotopy classes $[J]$ of almost complex structures on a 4-manifold M are compatible with a Kähler or symplectic structure. This is a very natural question about which little is known. In fact (apart from well known restrictions coming from characteristic classes) the first nontrivial example of a pair $(M, [J])$ with no symplectic realization is due to Taubes, who used his deep extension Seiberg–Witten theory. SW theory is still the only method available to tackle this kind of question.

Donaldson defined an involution on the set of free homotopy classes $[J]$ and proved some results about it for some Kähler manifolds. This paper extends these results to their natural range of applicability. The proofs are fairly straightforward applications of Donaldson's arguments. However, there are many cases to deal with, and the arguments are quite delicate.

I would not say that this paper has any really new idea. However, it is a useful contribution to the literature.

5. *Realizing homology classes by symplectic submanifolds.*

This paper again makes use of a different technique, namely Donaldson's method for constructing symplectic submanifolds that represent a class Poincaré dual to some cohomology class λc^k, where $c \in H^2(M;\mathbb{R})$ may be represented by a symplectic form and $\lambda > 0$.

The main result of this paper is that a multiple of *each* integral even dimensional homology class of M can be represented by the difference of two symplectic submanifolds. This is the sharpest result of this form that one could expect for arbitrary homology classes $[Z]$, since classes with symplectic representatives must be formally symplectic in the sense that there is some class of the form c^k that does not vanish on $[Z]$. The proof of this result is straightforward, but requires navigating quite the considerable technicalities of the work of Donaldson and Auroux, in particular.

In Thm 1.5, Le also applies Gromov's h-principle to prove the folk theorem that when $n \geq 3$ every manifold (M^{2n}, ω) such that $\omega|_{\pi_2(M)} \neq 0$ has a symplectically embedded 2-sphere.

As with 4. this carefully written paper does not contain any very new ideas but is a useful contribution.

6. *Weak equivalence classes of complex vector bundles.*

This paper gives the detailed proof of a result used in 5. about the existence of bundles with given Chern classes satisfying some restrictions on their classifying map. The argument uses purely classical algebraic topology but is nontrivial. The paper is carefully written and leads to some interesting questions in the category of holomorphic bundles.

Taken together these papers form an impressive collection. The first three contain the most substantial new ideas and make very significant contributions to Floer theory and Gromov–Witten theory, two of the most important techniques in symplectic topology today. Papers 4. and 5. show that Le has mastered the other main techniques in the subject, namely Taubes–Seiberg–Witten theory and Donaldson theory.

The papers show excellent scholarship and a wide breadth of knowledge and in my opinion certainly justify the granting of the Docent degree.

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