Silesian University in Opava
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Asymptotics of Bergman kernel and Berezin transform

Abstract of the Ph.D. Thesis
January 2014

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav v Opavě

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Asymptotika Bergmanova jádra a Berezinovy transformace

Autoreferát dizertační práce
Leden 2014

Matematická analýza

Výsledky uvedené v této dizertační práci byly získány v průběhu mého doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2009 - 2014. Výzkum byl podporován projekty MSM4781305904, SGS/17/2010, SGS/1/2013, IČ47813059 a také projektem "Podpora nadaných studentů a absolventů studia na SU - MSK".

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Školící pracoviště:	Matematický ústav Slezské univerzity v Opavě
Oponenti:	

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Státní doktorská zkouška a obhajoba dizertační práce se budou konat ...... před oborovou komisí doktorského studia Matematická analýza v zasedací místnosti rektorátu Slezské univerzity v Opavě.

Dizertační práce je dostupná v knihovně Matematického ústavu, Na Rybníčku 1, Opava.

Předseda Oborové rady:

The author would like to thank Prof. RNDr. Miroslav Engliš, DrSc. for his guidance and kind supervision over the years.

Contents

1. Structure of the thesis

The thesis is based on two papers [10], [11]. Their common subject is the notion of the Bergman kernel, which is briefly introduced along with its properties in Section 2. Proofs given in this section are common knowledge and can be found in any introductory work on the subject and they have to be credited to S. Bergman.

The first paper of the thesis [10] is dealing with the asymptotic behavior of Bergman kernel on the unit disc in complex plane as argument approaches the boundary with respect to a weight with a 'logarithmic singularity'. The motivation is to help in the quest of seeking a weighted analogue of the celebrated Fefferman's result about boundary behavior of Bergman kernel on strictly pseudoconvex domains. The literature concerning Fefferman's result is vast and its applications numerous. Originally, however, it serves the needs of Complex geometry in higher dimension – a subject which we briefly introduce in Section 3 even though the setting of the paper [10] is one dimensional. The exposition is based on the magnificent essays [24], [25], [6], [29] as well as on Fefferman's own works [23], [3].

The second paper [11] establishes an asymptotic expansion of the harmonic Berezin transform on the unit ball in $\mathbb{R}^n$. The result is in full analogy with the paper [22], and it is a generalization of the results obtained in [26], [27]. The asymptotic expansion of the *holomorphic* Berezin transform is of vital importance in the theory of Berezin-Toeplitz quantization on Kähler manifolds. We give a quite informal introduction to quantization problem and its relevance to the paper [11] in Section 4. For the inspiration we refer to the excellent surveys [21], [2].

Finally, we give more detailed information about results and methods of the papers [10] and [11] in Section 5.

2. Bergman kernels

Let Ω be domain in $\mathbb{C}$, that is an open connected *proper* subset of $\mathbb{C}$. Let $L^2(\Omega)$ be the Hilbert space of square-integrable functions, that is

$$f \in L^2(\Omega) \Leftrightarrow \|f\|^2 := \int_\Omega |f|^2 \, \mathrm{d}\lambda(z) < \infty,$$

where $\mathrm{d}\lambda(z)$ is the ordinary Lebesgue measure in $\mathbb{R}^2$. The scalar product is defined naturally as follows

$$\langle f, g \rangle := \int_\Omega f(z)\overline{g(z)} \mathrm{d}\lambda(z).$$

Consider now the *Bergman space* $A^2(\Omega)$ of all functions that are *holomorphic* on Ω and belong to $L^2(\Omega)$. It can be proved that A^2 is a closed subspace of $L^2(\Omega)$ (for example [30]) and therefore a Hilbert space in its own right. Moreover, the evaluation functional

$$e_x : f \longmapsto f(x)$$

is continuous there and thus by the Riesz representation theorem there exists a unique element K_x such that

$$\langle f, K_x \rangle = f(x), \qquad \forall f \in A^2(\Omega).$$

Denoting $K(x, z) := \overline{K_x(z)}$ we can therefore represent the point evaluation as an integral operator

$$f(x) = \int_\Omega f(z)K(x, z)\mathrm{d}\lambda(z), \tag{1}$$

with the kernel function $K(x, z)$ which is therefore suitably named the *Bergman kernel* and the equality (1) which holds for all $x \in \Omega$ and all $f \in A^2(\Omega)$ is the so-called *reproducing property*.

It is easy to see few basic properties of the Bergman kernel.

(1) $K(x,z)$ is holomorphic in x and in $\bar{z}$.
Anti-holomorphicity in z is clear form $K(x,z) = \overline{K_x(z)}$ and from the fact that $K_x(z)$ is holomorphic by definition of being the member of $A^2(D)$. The holomorphic behavior with respect to the x variable is a consequence of the next property.

(2) $K(x,z) = \overline{K(z,x)}$.
Since $\overline{K(z,\cdot)} = K_z(\cdot) \in A^2(\Omega)$ we can use the reproducing property to obtain

$$\begin{aligned} \overline{K(z,x)} &= \int_\Omega \overline{K(z,\xi)} K(x,\xi) \mathrm{d}\lambda(\xi) = \overline{\int_\Omega K(z,\xi)\overline{K(x,\xi)} \mathrm{d}\lambda(\xi)} \\ &= \overline{\overline{K(x,z)}} = K(x,z). \end{aligned}$$

(3) The Bergman kernel is unique. Indeed, lets consider two Bergman kernels $K_1(x,z)$ and $K_2(x,z)$ satisfying the properties (1),(2) above and the reproducing property. Then they are equal.

$$\begin{aligned} K_1(x,z) &= \int_\Omega K_1(\xi,z) K_2(x,\xi) \mathrm{d}\lambda(\xi) = \int_\Omega \overline{K_1(z,\xi)} K_2(x,\xi) \mathrm{d}\lambda(\xi) \\ &= \overline{\int_\Omega K_1(\xi,z) K_2(\xi,x) \mathrm{d}\lambda(\xi)} = \overline{K_2(z,x)} = K_2(x,z). \end{aligned}$$

Example 1. The Bergman Kernel for the unit disk $\Omega := \mathbb{D}$ takes form

$$K(x,z) = \frac{1}{\pi}\frac{1}{(1-\bar{z}x)^2}.$$

This can be actually computed quite directly. In the following let f be holomorphic in a neighborhood of $\overline{\mathbb{D}}$. From the Stokes theorem we have

$$\int_{\mathbb{D}} f(z)\partial_{\bar{z}} P(x,z,\bar{z}) \mathrm{d}\lambda(z) = \oint_{\partial\mathbb{D}} f(z) P(x,z,\frac{1}{z})\frac{\mathrm{d}z}{2\mathrm{i}}.$$

So we are looking for the function $P(x,z,\bar{z})$ with the property

$$P(x,z,\frac{1}{z}) = \frac{1}{\pi}\frac{1}{z-x}, \qquad z \in \partial\mathbb{D}, \tag{2}$$

for this granted we have by virtue of the Cauchy formula

$$\int_{\mathbb{D}} f(z)\partial_{\bar{z}} P(x,z,\bar{z}) \mathrm{d}\lambda(z) = \oint_{\partial\mathbb{D}} f(z) P(x,z,\frac{1}{z})\frac{\mathrm{d}z}{2\mathrm{i}} = \oint_{\partial\mathbb{D}} \frac{f(z)}{z-x}\frac{\mathrm{d}z}{2\pi\mathrm{i}} = f(x).$$

The function $k(x,z) := \partial_{\bar{z}} P(x,z,\bar{z})$ is in general the kernel of evaluation functional represented as an integral operator. There are quite number of such kernels however, for example the most trivial choice $P(x,z,\bar{z}) := \frac{1}{\pi(z-x)}$ leads to the Dirac delta function $k(x,z) = \delta(z-x)$ as is well known.

But in case of Bergman kernel there is additional requirement of anti-holomorphicity ($\partial_z K = 0$) which leads to a problem of solving the Laplace equation

$$\partial_z\partial_{\bar{z}} P = 0 \quad \text{on } \mathbb{D}, \qquad P|_{z\in\partial\mathbb{D}} = \frac{1}{\pi(z-x)}.$$

The solution is

$$P = \frac{\bar{z}}{\pi(1-\bar{z}x)},$$

and we get

$$k(x,z) = \partial_{\bar{z}} P = \frac{1}{\pi}\frac{1}{(1-\bar{z}x)^2}.$$

Since this function is clearly skew-symmetric and holomorphic in x, it is therefore the Bergman kernel by uniqueness. ⋆

The method used in the example (i.e. solving the corresponding Laplace equation) can be actually generalized for an arbitrary domain with smooth boundary (see [30, §2.3]). There is also a series representation for a Bergman Kernel.

Lemma 1. *Given orthonormal base $\{\varphi_k(z)\}_{k=1}^{\infty}$ of $A^2(\Omega)$ the Bergman kernel can be represented in terms of the infinite series*

$$K(x,z) = \sum_{k=0}^{\infty} \varphi_k(x)\overline{\varphi_k(z)}.$$

Proof.

$$\begin{aligned} K(x,z) &= \sum_{k=0}^{\infty} \langle K(\cdot,z), \varphi_k\rangle \varphi_k(x) = \sum_{k=0}^{\infty} \overline{\langle \varphi_k, K(\cdot,z)\rangle} \varphi_k(x) \\ &= \sum_{k=0}^{\infty} \overline{\langle \varphi_k, \overline{K(z,\cdot)}\rangle} \varphi_k(x) = \sum_{k=0}^{\infty} \overline{\langle \varphi_k, K_z\rangle} \varphi_k(x) = \sum_{k=0}^{\infty} \overline{\varphi_k(z)} \varphi_k(x). \end{aligned}$$

□

Perhaps most important property of Bergman kernels is their behavior under biholomorphical change of coordinates.

Lemma 2. *Let $\psi(z)$ be a biholomorphic mapping of the domain Ω_1 onto Ω_2 that is a bijective holomorphic map with holomorphic inverse. Then*

$$K_{\Omega_1}(x,z) = K_{\Omega_2}(\psi(x),\psi(z))\psi'(x)\overline{\psi'(z)}.$$

Proof. Let $\{\varphi_k(z)\}_{k=1}^{\infty}$ be a orthonormal system of the space $A^2(\Omega_2)$. It follows that $\{\varphi_k(\psi(z))\psi'(z)\}_{k=1}^{\infty}$ is the orthonormal system of the space $A^2(\Omega_1)$ since

$$\delta_j^k = \int\limits_{\Omega_2} \varphi_j(z)\overline{\varphi_k(z)} \mathrm{d}\lambda(z) = \int\limits_{\Omega_1} \varphi_j(\psi(z))\overline{\varphi_k(\psi(z))}\psi'(z)\overline{\psi'(z)} \mathrm{d}\lambda(z).$$

Thus

$$\begin{aligned} K_{\Omega_1}(x,z) &= \sum_{k=0}^{\infty} \overline{\varphi_k(\psi(z))\psi'(z)} \varphi_k(\psi(x))\psi'(x) = \sum_{k=0}^{\infty} \overline{\varphi_k(\psi(z))} \varphi_k(\psi(x))\psi'(x)\overline{\psi'(z)} \\ &= K_{\Omega_2}(\psi(x),\psi(z))\psi'(x)\overline{\psi'(z)}. \end{aligned}$$

□

Since the Riemann mapping theorem holds, that is every non-empty, open, simply connected, proper subset of $\mathbb{C}$ can be biholomorphicaly mapped onto the unit disc $\mathbb{D}$, this is (in theory) quite powerful result. In practice, however, concrete formulas for ψ are mostly unknown and even if they are known, usually they are not expressible in terms of elementary functions. There are few exceptions.

Example 2. The biholomorphic mapping which maps the upper half plane $U = \{z | \operatorname{Im} z > 0\}$ onto the unit disc is given by the Cayley transform

$$\psi(z) = \frac{z - \mathrm{i}}{1 - \mathrm{i}z},$$

which leads to the Bergman Kernel

$$K_U(x,z) = K_{\mathbb{D}}(\psi(x),\psi(z))\psi'(x)\overline{\psi'(z)} = -\frac{1}{\pi(x - \bar{z})^2}.$$

⋆

Example 3. The prototypical example of a 'complicated' biholomorphism mentioned widely in the literature (but mostly without details) is the one that bring square onto the unit disc. Square is a simple polygon and a biholomorphism that brings the upper half plane onto the interior of a simple polygon is the so-called Schwartz-Christoffel mapping [18]. For this case the mapping is the Incomplete elliptic integral of the first kind

$$\sigma(z) = \sqrt{2}F\left(\sqrt{z+1}; \frac{\sqrt{2}}{2}\right)$$

and the inverse (which we are interested in) takes form

$$\arcsin^2(\mathrm{sn}\frac{z}{\sqrt{2}}) - 1,$$

where sn(z) is the Jacobi elliptic function. This has to be further composed with the Möbius transform that maps $U \to \mathbb{D}$ to obtain the wanted biholomorphism. We will not attempt to compute the corresponding Bergman kernel. $\boxed{\star}$

In fact, it is often possible to go the opposite way and compute a biholomorphism $\psi(z)$ from the Bergman kernel as shown in the following Lemma:

Lemma 3. *Let $\psi(z)$ be a biholomorphism that maps a simply connected domain Ω onto the unit disc $\mathbb{D}$ such that $\psi(a) = 0$, $\psi'(a) > 0$ for some point $a \in \Omega$. Then it holds*

$$\psi'(z) = \sqrt{\frac{\pi}{K_\Omega(a,a)}}K_\Omega(z,a).$$

Proof. Using Lemma 2 we have

$$K_\Omega(a,a) = K_\mathbb{D}(\psi(a),\psi(a))\psi'(a)\overline{\psi'(a)} = K_\mathbb{D}(0,0)\psi'(a)^2 = \frac{1}{\pi}\psi'(a)^2,$$

thus

$$\psi'(a) = \sqrt{\pi K_\Omega(a,a)}.$$

Also we have

$$K_\Omega(z,a) = K_\mathbb{D}(\psi(z),\psi(a))\psi'(z)\overline{\psi'(a)} = K_\mathbb{D}(\psi(z),0)\psi'(z)\psi'(a) = \frac{1}{\pi}\psi'(z)\psi'(a),$$

therefore

$$\psi'(z) = \frac{\pi}{\psi'(a)}K_\Omega(z,a) = \sqrt{\frac{\pi}{K_\Omega(a,a)}}K_\Omega(z,a).$$

□

3. Higher dimension

3.1. **Pseudoconvex domains.** In the case of $\mathbb{C}$ the natural sets to consider are domains, i. e. open connected sets, for holomorphic functions cannot be (in general) extended to a larger domain. More precisely, to any boundary point $p \in \partial\Omega$ there exists a function holomorphic in Ω that cannot be analytically continued to any neighborhood of p. The function $\frac{1}{z-p}$ meets that requirement but in fact we can find a function for which the boundary is the so-called natural boundary, which means that it has a singularity in every point of a dense subset of the boundary.

In higher dimensions the situation is different. In 1906 Friedrich Hartogs found an example of open, connected set $H \subset \mathbb{C}^2$ such that for every holomorphic function on H there is a holomorphic extension to a strictly larger domain.

His example is actually quite simple:

$$H = \left\{(z_1,z_2) \in \mathbb{D}\times\mathbb{D}; |z_2| > \frac{1}{2}\right\} \cup \left\{(z_1,z_2) \in \mathbb{D}\times\mathbb{D}; |z_1| < \frac{1}{2}\right\}.$$

And functions holomorphic on this set can be continued to $\mathbb{D} \times \mathbb{D}$ by means of the Cauchy integral

$$\oint_{|\xi|=r} \frac{f(z_1, \xi)}{\xi - z_2} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}},$$

where $\frac{1}{2} < r < 1$.

This discovery led to the definition of *domain of holomorphy* – an open subset of $\mathbb{C}^n$ such that there exist a function holomorphic on it which cannot be extended to any of its boundary points.

Every domain in $\mathbb{C}$ is a domain of holomorphy and consequently every finite Cartesian product of such domains is. Also any convex domain of $\mathbb{C}^n$ is a domain of holomorphy and the property of being domain of holomorphy is moreover invariant under biholomorphism.

Most powerful description was given, however, by E. E. Levi in 1910 who found that the global property of being domain of holomorphy has a local consequences.

Theorem . *Let Ω be a domain of holomorphy, $p \in \partial\Omega$, U_p some neighborhood of p and r real C^2 function with $\mathrm{d}r \neq 0$ on U_p such that $\Omega \cap U_p = \{z \in \mathbb{C}^n; r(z) < 0\}$, then*

$$L_p(r,t) := \sum_{j,k=0}^{n} \frac{\partial^2 r(p)}{\partial z_j \partial \bar{z}_k} t_j \overline{t_k} \geq 0, \qquad \forall t \in \mathbb{C}^n : \sum_{j=0}^{n} \frac{\partial r(p)}{\partial z_j} t_j = 0. \tag{3}$$

In other words domains of holomoprhy (with smooth boundary) have the property that the complex Hessian $L_p(r,t)$ (or *Levi form*) of the defining function r at the boundary point p is positive semi-definite for all vectors t in the holomorphic tangent space to $\partial\Omega$ at p.

This property is remarkably similar to differential characterization of convex set since for any convex set that is given in the form $\{z \in \mathbb{C}^n; r(z) < 0\}$ with differentiable r it holds $L(r,t) > 0$ for all nonzero t tangent to $\partial\Omega$.

Domains that meet the condition (3) at every boundary point are therefore called *pseudoconvex*. And domains which satisfy (3) strictly, i.e. $L_p(r,t) > 0$ for all nonzero tangent t, are called *strictly pseudoconvex*.

Thus Levi's theorem can be restated that domains of holomorphy are pseudoconvex. It was long conjectured and finally proved in 1950's by K. Oka that the converse is also true – i.e. pseudoconvex domains are domains of holomorphy.

There are many characterizations of pseudoconvex domains, for example the domain Ω is pseudoconvex if and only if the function

$$\varphi(z) := -\ln(d(z, \partial\Omega))$$

is plurisubharmonic on Ω, where $d(z, \partial\Omega)$ means Euclidean distance from the boundary and plurisubharmonic means that the function restricted to any complex line where defined is subharmonic, i. e. $\Delta f \geq 0$ at every point.

Coincidently, for convex set, the function $\varphi(z)$ is a convex function.

Clearly, the unit ball $\mathbb{B}^{2n} \subset \mathbb{C}^n$ is strictly pseudoconvex. An example of the domain which is *weakly* (i.e. not strictly) pseudoconvex is the complex ellipsoid.

Example 4. The complex ellipsoid

$$E = \left\{(z,w) \in \mathbb{C}^2; |z|^2 + |w|^4 - 1 =: r(z,w) < 0\right\},$$

has the Levi form

$$L_{(z,w)}(r,t) = |t_1|^2 + 4\,|w|^2\,|t_2|^2 \geq 0,$$

so it is pseudoconvex. Consider the boundary point $w = 0, z = 1$. Vectors t tangent at that point are of the form $t = (0, t_2)$. Substituting this we see

$$L_{(z,w)}(r,t) = 0,$$

hence it is not strictly pseudoconvex. ⋆

Pseudoconvex domains are quite general and in most cases it is easier to work with strictly pseudoconvex ones – they possess, contrary to the general case, number of nice properties. For example, they are locally biholomorphicaly equivalent to convex domains.

The pseudoconvexity can be also alternatively defined via Bergman kernels as follows. Similarly, as in one-dimensional case, we can define for domain Ω the Bergman space $A^2(\Omega)$ of all holomoprhic functions that are square-integrable. Similarly, it can be proved that this space is in fact a Hilbert space in which the evaluation functional $e_z : f \mapsto f(z)$ is continuous. Therefore by the same construction we have the Bergman kernel $K(x, z)$.

The domain is pseudoconvex if and only if the function $K(z, z)$ tends to infinity as z approaches the boundary.

3.2. **Riemann mapping theorem.** Another difference between one dimensional and several dimensional case is that we loose the Riemann mapping theorem. It was shown by H. Poincaré in 1907 that even the two most simple domains – the unit ball $\mathbb{B}^2$ and the unit polydisc $P := \{(z, w) \in \mathbb{C}^2; |z| < 1, |w| < 1\}$ in $\mathbb{C}^2$ – are not biholomorphicaly equivalent.

His approach was based on the observation that the group of auto-biholomorphisms of every equivalent domains is the same. And he showed that in the case of the ball and the polydisc the groups differ.

Intuitively, however, we can understand their non-equivalence rather simply by observing that the unit ball has a smooth boundary while the polydisc has 'corners'.

On the other hand the unit ball and the ellipsoid E have both smooth boundaries and yet they too cannot be biholomorphicaly mapped onto each other. But their boundaries differ at the level of pseudoconvexity, since the ball is strictly pseudoconvex and the ellipsoid is not.

The main problem of Complex geometry is to determine when two given pseudoconvex domains Ω_1, Ω_2 are biholomorphicaly equivalent and, as we will see, boundary behavior plays crucial role.

It was realized by Poincaré that we can attach to the points on the boundary numbers that do not change under biholomorphism, providing therefore a powerful tool for the classification problem. If the corresponding number differs for two domains, there cannot be a biholomorphic map between them.

Existence of such numbers (or *invariants*) is in higher dimension assured by a simple counting argument: A smooth boundary of strictly pseudoconvex domain Ω can be locally in a neighborhood of $p \in \partial\Omega$ described as

$$\Omega \cap U_p = \{z \in \mathbb{C}^n; \mathrm{Re}(z_1) = f(\mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2), \ldots, \mathrm{Re}(z_n), \mathrm{Im}(z_n))\},$$

for f smooth. To the m-th order the boundary is hence given by an m-degree polynomial of $2n - 1$ real variables – there are thus $\begin{pmatrix} m + 2n - 1 \\ 2n - 1 \end{pmatrix}$ real coefficients to specify. On the other hand the m-th order Taylor expansion around p of any biholomorphism is given by at most n polynomials of just n real variables (because of analyticity) – that is $n \begin{pmatrix} m + n \\ n \end{pmatrix}$ complex or $2n \begin{pmatrix} m + n \\ n \end{pmatrix}$ real coefficients. For $n \geq 2$ we can see that the first number is eventually greater than the second one as m grows. In other words there are more smooth boundaries than there are biholomorphic maps smooth up to the boundary, hence there must be many 'things' that do not change (a complete characterization of those 'things' was given by Chern and Moser [13] and are called Chern-Moser invariants).

The crucial point, however, is whether a biholomorphic map must be smooth up to the boundary.

In one dimension, the answer is yes. A biholomoprhic map between two bounded domains $\psi : \Omega_1 \to \Omega_2$ with smooth boundaries can be, indeed, extended to C^∞ diffeomorphism of the closures $\tilde{\psi} : \overline{\Omega}_1 \to \overline{\Omega}_2$ – a result which was proven first by Painlevé in 1887 [28]. But at the time of Poincaré nobody knew if the same thing is true also in higher dimensions.

The long standing conjecture was finally proven in 1974 in case of strictly pseudoconvex domains with C^∞ boundary by Charles Fefferman [23] who received his Fields medal for it. The main idea of his proof was to study the boundary behavior of geodesics in Bergman metric which in turn requires knowledge about asymptotic behavior of Bergman kernel. Fefferman proved the following theorem:

Theorem 1. *Let Ω be a strictly pseudoconvex domain in the form $\Omega = \{z \in \mathbb{C}^n; \psi(z) > 0\}$, where $\psi \in C^\infty$ is a real function with $d\psi \neq 0$ on the boundary, then*

$$K_\Omega(x,x) = \frac{\varphi(x)}{\psi(x)^{n+1}} + \tilde{\varphi}(x) \ln \psi(x),$$

where $\varphi, \tilde{\varphi}$ are smooth on the closure $\overline{\Omega}$, φ nonvanishing.

The result was generalized outside the diagonal by Boutet de Monvel and Sjöstrand in [12], that is

$$K_\Omega(x,z) = \frac{\varphi(x,z)}{\psi(x,z)^{n+1}} + \tilde{\varphi}(x,z) \ln \psi(x,z),$$

by using extensions for $\varphi, \psi, \tilde{\varphi}$.

The extensions must be understood as follows: For every C^∞ function $\psi(z)$ there exists a function $\psi(x,z)$ for which $\overline{\partial_x}\psi$ and $\partial_z\psi$ vanish to infinite order at the diagonal $x = z$ and $\psi(z,z) = \psi(z)$. Such an extension is known to always exists and is unique up to functions which are zero to infinite order at the diagonal.

It turns out that the boundary behavior of Bergman kernel is an interesting subject in its own right and has many application even outside Complex geometry, for example the asymptotic formula is used in study of asymptotic expansion of the so-called Berezin transform which plays the crucial role in theory of quantization on Kähler manifolds – a subject which we briefly introduce in the next section.

3.3. **Weighted Bergman spaces.** The first paper of the thesis [10] helps to understand (by computing a specific example) a natural question of whether there is some analogue of Fefferman's result in the setting of *weighted* Bergman space.

Weighted spaces are obtained quite simply by adding the term – a positive integrable 'weight' function w – to the volume form in the definition of the inner product so that it takes the form

$$\langle f, g \rangle := \int_\Omega f(z)\overline{g(z)} w(z) \mathrm{d}\lambda(z).$$

The Bergman space is then a set of functions, holomorphic on Ω such that the corresponding norm $\|f\| := \sqrt{\langle f, f\rangle}$ is finite. The reproducing kernel exists and it is derived as in Section 2.

The boundary behavior of this kernel with respect to non trivial weights is, unfortunately, in general unknown, there is, however, an analogue of Fefferman's result for the weights of the form

$$w = \psi^\alpha e^g, \qquad \alpha > -1 \tag{4}$$

where ψ is the defining function of a bounded, strictly pseudoconvex domain $\Omega = \{z \in \mathbb{C}^n; \psi(z) > 0\}$ and g is a smooth function up to the boundary. More concretely, it holds

$$K_\Omega(x,z) = \frac{\varphi(x,z)}{\psi(x,z)^{n+1} w(x,z)} + \tilde{\varphi}(x,z) \ln(\psi(x,z)) + \varphi_2(x,z), \tag{5}$$

where $\varphi, \tilde{\varphi}, \varphi_2$ are some smooth functions on the closure, φ nonvanishing and the extended weight $w(x,z)$ is produced by extending ψ and g as in Fefferman's theroem.

This with much more details was shown in [20] by M. Engliš. Later the same author generalized the result [19] also to weights of the form

$$w \approx \psi^\alpha e^g \left(1 + \sum_j \psi^{\alpha_j} \left(\ln \frac{1}{\psi}\right)^{\beta_j} g_j\right),$$

where α_j is a sequence of positive real numbers with limit $+\infty$, g_j are smooth functions on the closure of the domain and β_j are real numbers. In other words, they are weights whose main term has the form (4), but there are 'logarithmic' singularities allowed in terms of higher order.

The thesis paper [10] deals with the case when there is a logarithmic singularity also in the main term – though only for the most simple case of radially symmetric weights in the unit disc $\mathbb{D}$ (i.e. $\psi(z) = 1 - |z|^2$) in one dimension.

More precisely, the weights are of the form

$$w(z) \approx \left(1-|z|^2\right)^\alpha \sum_{k=0}^{\infty} w_k \ln^{\beta-k} \frac{1}{1-|z|^2}, \qquad (z \to 1),$$

where $\alpha > -1$, $\beta \in \mathbb{R}$, $w_0 = \frac{1}{\pi}$. And the main result (translated to our notation) is[1]

$$K(x,z) \sim \frac{\alpha+1}{(1-x\bar{z})^{\alpha+2} \ln^\beta \frac{1}{1-x\bar{z}}}, \qquad (x\bar{z} \to 1),$$

in other words

$$K(x,z) \sim \frac{\alpha+1}{\pi \psi(x,z)^2 w(x,z)}, \qquad (x\bar{z} \to 1),$$

so the principal term of Bergman kernel is still the same as the principal term in (5). On the other hand, the full asymptotic expansion we give, is in negative powers of $-\ln\psi$, that is

$$\frac{(1-x\bar{z})^{\alpha+2} \ln^\beta \frac{1}{1-x\bar{z}}}{\alpha+1} K(x,z) \approx 1 + \frac{d_1}{\ln \frac{1}{1-x\bar{z}}} + \frac{d_2}{\ln^2 \frac{1}{1-x\bar{z}}} + \dots, \quad (x\bar{z} \to 1),$$

where the coefficients d_k (given by a recurrent formula) are such that infinitely of them are non-zero even when the expansion of w contains only one term, so quite different behavior as in (5).

This result was obtained by ad hoc 'elementary' means which do not mimic those of works [20],[19], [23]. Obviously, the next logical step would be to somehow generalize this result to an arbitrary domain in $\mathbb{C}$ and then to an arbitrary strictly pseudoconvex domain in $\mathbb{C}^n$.

4. Berezin transform

4.1. **Quantization.** The quantum theory aspires to explain the apparent strangeness of the physical laws at the microscopic level by introducing some non-commutativity to their mathematical description. It was long ago realized that the so-called Heisenberg uncertainty principle and the fact that certain measurements are 'quantized' – i.e. the measured values seem to come up only as integer multiples of a unit – this all can be explained mathematically if one replaces functions f (i.e. classical observables) by some operators Q_f (quantum observables).

Non-commutativity of such operators implies Heisenberg uncertainty principle and measured values are defined to be only members of their spectra – which can easily be a discrete set. Since the result of any experiment is a real number, usually, there is an additional requirement that the spectrum of any operator Q_f is subset of a real line (for real symbol f).

Consider a function $f(p,q)$ of position $q \in \mathbb{R}$ and momentum $p \in \mathbb{R}$ which describes classical evolution of a particle in the Hamilton formalism. In *canonical quantization* one tries to assigns to every such f a self-adjoint operator Q_f on the Hilbert space $L^2(\mathbb{R})$ such that

$$\begin{aligned} Q_q &= q, \qquad \text{i.e. multiplication operator,} \\ Q_p &= -\mathrm{i}\hbar\partial_q, \end{aligned}$$

where $\hbar$ is the reduced Planck constant $\hbar = \frac{h}{2\pi}$. The fact that

$$[Q_q, Q_p] = \mathrm{i}\hbar I,$$

implies Heisenberg uncertainty principle. Note that when $\hbar \searrow 0$ (or as observer's perspective gets larger and larger) the non-commutativity disappears, hence the classical physics is recovered. This idea originated in works of Weyl, von Neumann and Dirac.

The main problem is how to choose the assignment $f \to Q_f$ for more general functions than the coordinate ones. From physical point of view one would like that $f \to Q_f$ fulfills following properties

(1) $f \to Q_f$ is linear,
(2) for any polynomial r, it holds $Q_{r(f)} = r(Q_f)$ (von Neumann rule),
(3) $[Q_f, Q_g] = -\mathrm{i}\hbar Q_{\{f,g\}}$ (Commutation relation),

[1] Here throughout the thesis the symbol $\sim$ means $f \sim g \Leftrightarrow f/g \to 1$ while $\approx$ stands for asymptotic expansion.

where $\{f,g\} := \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$ is Poisson bracket.

Unfortunately, these properties are mutually inconsistent to a spectacular degree. Not only all three combined, but also any two of them are inconsistent (for references see [21]). It can be even shown that von Neumann rule *alone* leads to a contradiction if we require it to hold for too wild functions (not just polynomials) like the Peáno curve (again see [21]). In fact, every mathematical approach that tries to remedy these inconsistencies starts with cutting down von Neumann rule to a bare minimum $Q_1 = I$.

In the following we will consider the so-called *deformation quantization*, that is we keep the linearity of the assignment $f \to Q_f$, discard von Neumann rule except $Q_1 = I$ and we relax Commutation relation property to hold only asymptotically as $\hbar \searrow 0$, that is

$$[Q_f, Q_g] = -\mathrm{i}\hbar Q_{\{f,g\}} + O(\hbar^2), \qquad (\hbar \searrow 0).$$

In the computation of the commutator $[\cdot,\cdot]$ one usually tries to expand the product $Q_f Q_g$ first. Obviously the first term has to be

$$Q_f Q_g \sim Q_{fg}, \qquad (\hbar \searrow 0),$$

otherwise the classical physics cannot be restored. Then we compute the next term

$$Q_f Q_g - Q_{fg} \sim \hbar Q_{C_1(f,g)}, \qquad (\hbar \searrow 0),$$

where $C_1(f,g)$ is some bilinear operator. If it happens that

$$C_1(f,g) - C_1(g,f) = -\mathrm{i}\,\{f,g\}\,,$$

then the commutation relation is fulfilled. We can go on and produce the next term

$$Q_f Q_g - Q_{fg} - \hbar Q_{C_1(f,g)} \sim \hbar^2 Q_{C_2(f,g)}, \qquad (\hbar \searrow 0),$$

and the next one and so on until we have the full expansion

$$Q_f Q_g \approx \sum_{k=0}^{\infty} \hbar^k Q_{C_k(f,g)}, \qquad (\hbar \searrow 0).$$

We can summarize this by defining the so-called *star product* $*$ in the following way

$$Q_f Q_g = Q_{f*g},$$

where

$$f * g = \sum_{k=0}^{\infty} \hbar^k C_k(f,g). \tag{6}$$

The series must be understood only as a formal power series in $\hbar$ since no convergence is assured. Everything is usually presented the other way around and the series (6) is taken as a definition of the star product together with the condition that the bilinear operators $C_k(f,g)$ must satisfy the following properties

$$C_0(f,g) = fg, \qquad C_1(f,g) - C_1(g,f) = -\mathrm{i}\,\{f,g\}\,, \qquad C_k(1,f) = C_k(f,1) = 0 \quad \forall k > 0.$$

One also requires that such a star product $*$ is an associative operation. With such a definition one can do quantization without the underlying Hilbert space and operator calculus and work solely in terms of star product. This is at least the main idea behind *formal deformation quantization* introduced in work [5] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer.

On the other hand, it is desirable to keep the operator picture in sight since with formal power series it is difficult to perform concrete computations. One can easily encounter series which converge for no value of $\hbar$ (except for $\hbar = 0$) and so on. We will therefore keep focus on operators, though, admittedly, without subtleties about their domains of definitions and convergence issues. Most of the time we will consider for symbols f only polynomials and just slightly touch the definition for more general symbols.

There are number of ways how to get deformation quantization. Intuitively, the problem rises with *choice of ordering*. We are trying to replace something commutative with something which is not. In commutative realm we cannot distinguish between pq and qp, while in the operator case we sure can $-\mathrm{i}\hbar\partial_q q \neq -\mathrm{i}\hbar q \partial_q$. So a polynomial $g(p,q)$ cannot be readily replaced with $g(-\mathrm{i}\hbar\partial_q, q)$ without specifying some order of operations.

There are many ways how to specify the order (indeed, infinitely many), most of the attention (as far as the present author can see) lies with the following three.

4.2. **Pseudo-differential operators.** When all derivatives are taken to the right (so called *anti-Wick* ordering) this leads to the definition of *pseudo-differential operators.* By symbol

$$\overrightarrow{g}(A,B) := \sum_{j,k=0}^{\infty} \frac{\partial_1^k \partial_2^j g(0,0)}{j!k!} A^k B^j,$$

we mean that second argument always takes precedent over the first one. In our case, to a polynomial $g(q,p)$ is assigned the operator $\overrightarrow{g}$

$$\overrightarrow{g}(q,-\mathrm{i}\hbar\partial_q) := \sum_{k=0}^{\infty} \frac{\partial_2^k g(q,0)}{k!}(-\mathrm{i}\hbar\partial_q)^k.$$

Observe that on exponentials the action is computed

$$\overrightarrow{g}(q,-\mathrm{i}\hbar\partial_q)e^{\alpha q} = g(q,-\mathrm{i}\hbar\alpha)e^{\alpha q},$$

for all $\alpha \in \mathbb{C}$. And if we can expand the function f into exponentials via Fourier transform and its inverse we get

$$\overrightarrow{g}(q,-\mathrm{i}\hbar\partial_q)f(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) g(q,\hbar\xi) e^{\mathrm{i}\xi q} \mathrm{d}\xi := \Psi_g f.$$

Since the integral representation is more general it usually serves as a definition of pseudo-differential operator Ψ_g.

Now let $f(q,p) := \sum_{k=0}^{\infty} f_k(q)p^k$ and $q(q,p) := \sum_{k=0}^{\infty} g_k(q)p^k$ some polynomials of unspecified degree. We have

$$\begin{aligned}\overrightarrow{g}(q,-\mathrm{i}\hbar\partial_q)\overrightarrow{f}(q,-\mathrm{i}\hbar\partial_q) &= \left(g_0 - \mathrm{i}\hbar g_1\partial_q + O(\hbar^2)\right)\left(f_0 - \mathrm{i}\hbar f_1\partial_q + O(\hbar^2)\right), \\ &= g_0 f_0 - \mathrm{i}\hbar\left(f_0 g_1 \partial_q + f_1 g_0 \partial_q + f_1 g_0'\right) + O(\hbar^2).\end{aligned}$$

Thus

$$\overrightarrow{g}\overrightarrow{f} - \overrightarrow{f}\overrightarrow{g} = -\mathrm{i}\hbar\left(f_1 g_0' - g_1 f_0'\right) + O(\hbar^2) = -\mathrm{i}\hbar\overrightarrow{\{g,f\}} + O(\hbar^2).$$

So, indeed, the asymptotical commutation relation (at least for polynomials) is fulfilled.

Equating the coefficients at the same power of $\hbar$ in the equality

$$\overrightarrow{g}(q,-\mathrm{i}\hbar\partial_q)\overrightarrow{f}(q,-\mathrm{i}\hbar\partial_q) = \sum_{k=0}^{\infty} \hbar^k \overrightarrow{C_k(g,f)}(q,-\mathrm{i}\hbar\partial_q),$$

we get

$$C_k(g,f)(q,p) := \frac{(-\mathrm{i})^k}{k!}(\partial_p^k f(q,p))(\partial_q^k g(q,p)).$$

Let us define

$$f *_\psi g = \sum_{k=0}^{\infty} \frac{(-\mathrm{i}\hbar)^k}{k!}(\partial_p^k f(q,p))(\partial_q^k g(q,p)).$$

We can easily check that such a formal power series satisfies all the necessary properties required for a star product.

So even though pseudo-differential operators cannot serve as an example for deformation quantization since they are not in general self-adjoint – in fact

$$\overrightarrow{r}(q,-\mathrm{i}\hbar\partial_q)^\dagger = \overrightarrow{r}(-\mathrm{i}\hbar\partial_q,q),$$

there *is* a good star product that goes with them.

4.3. **Weyl calculus.** The one ordering that does produce self-adjoint operators is the so-called *Weyl calculus*. It is a 'fair' ordering, where all orderings are considered simultaneously by taking their arithmetic mean. For example $pq \to -\frac{\mathrm{i}\hbar}{2}(q\partial_q + \partial_q q)$, $q^2 p \to -\frac{\mathrm{i}\hbar}{3}\left(q^2\partial_q + q\partial_q q + \partial_q q^2\right)$ and so on. We will denote an operator with such ordering $\overleftrightarrow{g}(q, -\mathrm{i}\hbar\partial_q)$.

To get an assignment for an arbitrary monomial $q^k p^l$ one can expand the expression $(A+B)^{k+l}$ for some non-commutative A, B and take those term containing exactly k A-s and l B-s and then substitute $A = q$ and $B = -\mathrm{i}\hbar\partial_q$. There will be precisely $\begin{pmatrix} k+l \\ l \end{pmatrix}$ such terms hence one should divide by this number.

From this one can see, that for polynomials of a special form $(rp + sq)^k$ the assignment is simply

$$(sq + rp)^k \to (sq - \mathrm{i}\hbar r\partial_q)^k.$$

Hence for any polynomial g of one variable we have

$$g(sq + rp) \to g(sq - \mathrm{i}\hbar r\partial_q),$$

and it is not difficult to see that this can be computed as an ordinary differential operator, since it holds

$$g(sq - \mathrm{i}\hbar r\partial_q) = e^{-\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}} g(-\mathrm{i}\hbar r\partial_q) e^{\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}}. \tag{7}$$

By analogy we can define the assignment further to entire functions, exponentials especially to get

$$e^{sq+rp} \to e^{-\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}} e^{-\mathrm{i}\hbar r\partial_q} e^{\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}},$$

when acted on polynomials the action is well defined and moreover easily computed, since on polynomials f the exponential is clearly acting as a translation

$$e^{-\mathrm{i}\hbar r\partial_q} f(q) = f(q - \mathrm{i}\hbar r),$$

We hence define

$$e^{sq - \mathrm{i}\hbar r\partial_q} f(q) = e^{-\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}} e^{-\mathrm{i}\hbar r\partial_q} e^{\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}} f(q) = e^{-\mathrm{i}\frac{s}{\hbar r}\frac{q^2}{2}} e^{\mathrm{i}\frac{s}{\hbar r}\frac{(q-\mathrm{i}\hbar r)^2}{2}} f(q - \mathrm{i}\hbar r),$$

thus

$$e^{sq - \mathrm{i}\hbar r\partial_q} f(q) := e^{sq - \frac{1}{2}\mathrm{i}\hbar sr} f(q - \mathrm{i}\hbar r).$$

Finally, an 'arbitrary' function $g(q,p)$ can be represented as an exponential of this kind via Fourier transform

$$g(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathcal{F}_1\mathcal{F}_2 g)(\xi_1, \xi_2) e^{\mathrm{i}\xi_1 q + \mathrm{i}\xi_2 p} \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$

hence we have (for $s = \mathrm{i}\xi_1, r = \mathrm{i}\xi_2$)

$$\begin{aligned}
\overleftrightarrow{g}(q, -\mathrm{i}\hbar\partial_q) f(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathcal{F}_1\mathcal{F}_2 g)(\xi_1, \xi_2) e^{\mathrm{i}\xi_1 q + \frac{1}{2}\mathrm{i}\hbar\xi_1\xi_2} f(q + \hbar\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\mathcal{F}_2 g)(q + \frac{1}{2}\hbar\xi_2, \xi_2) f(q + \hbar\xi_2) \mathrm{d}\xi_2 \\
&= \frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty} (\mathcal{F}_2 g)\left(\frac{\xi_2 + q}{2}, \frac{\xi_2}{\hbar}\right) f(\xi_2) \mathrm{d}\xi_2 \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(\frac{\xi + q}{2}, p\right) f(\xi) e^{-\mathrm{i}p\frac{\xi}{\hbar}} \mathrm{d}p\mathrm{d}\xi =: W_g f.
\end{aligned}$$

With same technique as in previous section we can check that Commutation relation is indeed satisfied (at least for polynomials). Also one can proceed further and compute terms of the associated star product

$*_W$ but they are quite complicated and we will not endeavor to derive them here. Nevertheless, the Weyl calculus constitutes an example of deformation quantization.

4.4. **Toeplitz operators.** Last interesting ordering is when all derivatives are taken to the left (the so-called *Wick ordering*):

$$\overleftarrow{g}(A, B) := \sum_{k=0}^{\infty} B^k \frac{\partial_2^k g(A, 0)}{k!},$$

Consider a polynomial $g(z, \bar{z})$ in complex coordinates. The action of an operator $\overleftarrow{g}(z, 1/\alpha\partial_z)$ can be computed as follows:

$$\overleftarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right) f(z) = \sum_{k=0}^{\infty} \frac{\partial_z^k \partial_{\bar{x}}^k}{\alpha^k k!} g(z, \bar{x}) f(z)|_{x=0} = \sum_{k=0}^{\infty} \frac{\partial_x^k \partial_{\bar{x}}^k}{\alpha^k k!} g(z + x, \bar{x}) f(z + x)|_{x=0}$$

$$e^{\frac{\partial_x \partial_{\bar{x}}}{\alpha}} g(z + x, \bar{x}) f(z + x)|_{x=0}.$$

Remmeber, that the operator $e^{\frac{\partial_x \partial_{\bar{x}}}{\alpha}}$ is the heat equation operator, i.e. the expression above is the solution of the equation

$$u_t = \frac{1}{4}\Delta_x u, \qquad u|_{t=0} = g(z + x, \bar{x}) f(z + x),$$

at the time $t = \frac{1}{\alpha}$. Substituting into the fundamental solution of the heat equation and letting $x = 0$ we thus get

$$\overleftarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right) f(z) = \int_{\mathbb{C}} g(x + z, \bar{x}) f(x + z) e^{-\alpha x \bar{x}} \frac{\alpha}{\pi} \mathrm{d}\lambda(x)$$

$$= \int_{\mathbb{C}} g(x, \bar{x}) f(x) e^{\alpha \bar{x} z} \mathrm{d}\mu_\alpha(x, \bar{x}) =: T_g f, \tag{8}$$

where $\mathrm{d}\mu_\alpha(x, \bar{x}) := e^{-\alpha x \bar{x}} \frac{\alpha}{\pi} \mathrm{d}\lambda(x)$.

Again, the last integral is usually taken as a definition of the so-called *Toeplitz operator* T_g acting on f.

Toeplitz operators have the advantage over the Weyl calculus that they can be readily extended to a more general domains in $\mathbb{C}$ (and $\mathbb{C}^n$) since they not require the notion of Fourier transform which is exclusive property of Euclidean space and few other spaces.

What they do in fact require is the notions of (weighted) Bergman kernel and the (weighted) Bergman projection which are both definable in arbitrary domains.

Why this is so? The space $\mathcal{F}_\alpha$ of all entire functions in $\mathbb{C}$ that are square integrable with respect to the measure $\mathrm{d}\mu_\alpha(x, \bar{x})$ forms the weighted Bergman space (it is a Hilbert space with continuous evaluation functional). Associated Bergman kernel introduced in Section 2 takes form

$$K_\alpha(z, x) = e^{\alpha z \bar{x}}.$$

Remember that Bergman kernel satisfies the reproducing property

$$f(z) = \int_{\mathbb{C}} f(x) K_\alpha(z, x) \mathrm{d}\mu_\alpha(x, \bar{x}), \qquad \forall f \in \mathcal{F}_\alpha.$$

The last integral, however, can be applied to a more general functions than just members of $\mathcal{F}_\alpha$. Denote

$$(P_\alpha f)(z) := \int_{\mathbb{C}} f(x, \bar{x}) K_\alpha(z, x) \mathrm{d}\mu_\alpha(x, \bar{x}).$$

Clearly, the result for a function f from $L^2(\mathbb{C}, \mathrm{d}\mu_\alpha)$ is an entire function, hence by reproducing property we have $P_\alpha^2 f = P_\alpha f$, thus the operation $P_\alpha f$ constitute a projection and is named *Bergman projection.* With these definitions in hand we see, that the action of Toeplitz operators (8) can be written

$$T_g f = (P_\alpha g f)(z).$$

This 'coordinate free' definition makes sense in all Bergman spaces. With bounded symbols g, the operator T_g is even continuous with $\|T_g\| \leq \|g\|_\infty$.

But to obtain a deformation quantization from this, one has to face some obstacles first. The direct approach of assigning $g(q,p) \to \overleftarrow{g}(q, -\mathrm{i}\hbar\partial_q)$ is no good since α is in this case purely imaginary and we do not get a particularly rich Bergman space – $\mathcal{F}_\alpha$ would actually contain only the zero function. Also, operators $\overleftarrow{g}(q, -\mathrm{i}\hbar\partial_q)$ suffer the same condition as pseudo-differential operators of being not self-adjoint. In fact, those two operators are dual to each other.

The standard workaround is to first transform a polynomial $g(q,p)$ into complex coordinates $g(q - \mathrm{i}p, q + \mathrm{i}p)$ and then assign

$$\overleftarrow{g}(q - \hbar\partial_q, q + \hbar\partial_q).$$

Clearly, this is self-adjoint:

$$\overleftarrow{g}(q - \hbar\partial_q, q + \hbar\partial_q)^\dagger = \overrightarrow{g}(q + \hbar\partial_q, q - \hbar\partial_q) = \overleftarrow{g}(q - \hbar\partial_q, q + \hbar\partial_q).$$

This neat trick actually works for every ordering not just Wick ordering.

Consider now the so-called *Bargmann transform*

$$(\beta f)(z) := \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} f(q) e^{-\frac{1}{2\hbar}q^2 + \frac{1}{\hbar}qz - \frac{1}{4\hbar}z^2} \mathrm{d}q,$$

which is a unitary isomorphism between $L^2(\mathbb{R})$ and the Fock space $\mathcal{F}_\alpha$ with $\alpha = \frac{1}{2\hbar}$. It is an easy exercise to show that Bargmann transform posses the following properties

$$\begin{aligned} 2\hbar\partial_z(\beta f)(z) &= (\beta(q + \hbar\partial_q)f)(z), \\ z(\beta f)(z) &= (\beta(q - \hbar\partial_q)f)(z). \end{aligned}$$

So we can see that

$$\overleftarrow{g}(q - \hbar\partial_q, q + \hbar\partial_q)f(q) = \beta^{-1}\overleftarrow{g}(z, 2\hbar\partial_z)(\beta f)(z) = \beta^{-1}T_g(\beta f)(z),$$

where the Toeplizt operator T_g depends also on parameter $\alpha = \frac{1}{2\hbar}$ even though it is not explicitly mentioned. It is better in this setting to consider the Fock space $\mathcal{F}_\alpha$ rather then $L^2(\mathbb{R})$ space as our Hilbert space and to move from p, q coordinates to $z, \bar{z}$ coordinates. Deformation quantization problem translates in this notation to a problem of assigning to functions $f(z, \bar{z})$ operators (not necessarily self-adjoint) Q_f on $\mathcal{F}_\alpha$ with $\alpha = \frac{1}{2\hbar}$ such that

(1) $Q_z = z$, $Q_{\bar{z}} = \frac{1}{\alpha}\partial_z$, $Q_1 = I$.
(2) $f \to Q_f$ is linear.
(3) $[Q_f, Q_g] \sim \frac{1}{\alpha}Q_{\{f,g\}}, \qquad (\alpha \to \infty)$,

where the Poisson bracket is defined $\{f, g\} := \partial_{\bar{z}} f \partial_z g - \partial_{\bar{z}} g \partial_z f$.

We can see that case $Q_f := T_f$ fulfills these requirements since in case of polynomial symbol $f = g(z, \bar{z})$ we still have

$$T_g = \overleftarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right),$$

and the complex commutation relation (3) can be verified directly (for polynomials) as in the section about pseudo-differential operators. The associated star product is almost the same, namely

$$f *_T g = \sum_{k=0}^{\infty} \frac{(-1)^j}{\alpha^j j!} (\overline{\partial}^k g)(\partial^k f).$$

This is, actually, a consequence of the fact that Toeplitz operators are adjoint to the pseudodifferential ones

$$\overleftarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right)^\dagger = \overrightarrow{g}\left(z, -\frac{1}{\alpha}\partial_z\right).$$

Generally, if there is a star product defined

$$Q_f Q_g = Q_{f*g},$$

the star product for the adjoint operators

$$Q_f^\dagger Q_g^\dagger = Q_{f*_\dagger g}^\dagger,$$

takes form

$$f *_\dagger g = g * f,$$

which can be seen from the equality

$$Q_{f*g}^\dagger = (Q_f Q_g)^\dagger = Q_g^\dagger Q_f^\dagger = Q_{g*_\dagger f}^\dagger.$$

Hence

$$f *_T g = g *_\psi f,$$

as claimed.

Analogously, we can define even pseudo-differential operators and Weyl calculus in this setting by the following rules

$$\begin{aligned} \overrightarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right) f(z) &= \int_{\mathbb{C}} f(x) g(z, \bar{x}) e^{\alpha \bar{x} z} \mathrm{d}\mu_\alpha(x, \bar{x}), \\ \overleftrightarrow{g}\left(z, \frac{1}{\alpha}\partial_z\right) f(z) &= \int_{\mathbb{C}} f(x) g\left(\frac{x+z}{2}, \bar{x}\right) e^{\alpha \bar{x} z} \mathrm{d}\mu_\alpha(x, \bar{x}). \end{aligned}$$

4.5. **Berezin transform.** As been said before, the Topelitz quantization (that is the assignment $f \to T_f$) is best suited (from given examples at least) to be generalizable to other domains in $\mathbb{C}$, since the Toeplitz operator T_g can be defined on arbitrary weighted Bergman space $A^2(\Omega)$ – i.e. the space of all holomorphic function on Ω such that

$$\|f\|^2 := \int_\Omega |f(x)|^2 w(x) \mathrm{d}\lambda(x) < \infty,$$

in an intrinsic way, concretely

$$T_g f := P(gf), \qquad \forall f \in A^2(\Omega),$$

where P is the corresponding Bergman projection, that is

$$(Pf)(z) = \int_\Omega f(x) K(z, x) w(x) \mathrm{d}\lambda(x),$$

and $K(z, x)$ is Bergman kernel. What is unclear, however, is how to choose the weight function w such that the complex commutation relation is fulfilled. In the case $\Omega = \mathbb{C}$ the weight turned out to be $w(x) = e^{-\alpha x \bar{x}} \frac{\alpha}{\pi}$.

Let our domain admit a defining function ψ that is $\Omega = \{x \in \mathbb{C} : \psi(x) > 0\}$ with $\mathrm{d}\psi \neq 0$ on the boundary $\partial\Omega$. Let on a neighborhood of $\overline{\Omega}$ also holds

$$\partial_x \partial_{\bar{x}} \ln \frac{1}{\psi} > 0,$$

in other words the function $-\ln\psi$ is strictly subharmonic.

It can bee proved (see [21, §4.7]) that under such conditions the correct weight can be chosen essentially as a power of the defining function, i. e.

$$w = c_\alpha \psi^\alpha \partial\overline{\partial} \ln\psi,$$

where c_α is some constant to ensure that the corresponding measure $w(x)\mathrm{d}\lambda(x)$ is a probabilistic one (otherwise we cannot have $T_1 = I$). In the case $\Omega = \mathbb{C}$ the most simple function that satisfies the strict subharmonicity condition is $-\ln\psi = |x|^2$, $-\partial_x \partial_{\bar{x}} \ln\psi = 1$ and we can see that $\psi = e^{-|x|^2}$ is a defining function for $\mathbb{C}$ and the corresponding weight is $w(x) = c_\alpha e^{-\alpha|x|^2}$ as promised. In the case of the unit disk $\Omega = \mathbb{D}$, the defining function is $\psi = 1 - |x|^2$, so we have

$$w(x) = c_\alpha \left(1 - |x|^2\right)^{\alpha - 2}.$$

The Poisson bracket for the domain Ω is defined

$$\{f,g\} := -\frac{1}{\partial_{\bar z}\partial_z \ln\psi}\left(\partial_{\bar z} f \partial_z g - \partial_{\bar z} g \partial_z f\right).$$

It is easy to see that the defining function ψ we require always exists in case of one dimension (for bounded domains we can take $\psi(z)$ to be a smooth function that coincides in a neighborhood of the boundary with the Euclidean distance of z to the boundary). In higher dimension the condition that $-\partial_x\partial_{\bar x}\ln\psi > 0$ is replaced by the condition that the function $-\ln\psi$ is strictly plurisubharmonic – in other words we require that our domain is strictly pseudoconvex.

The correction factor for the Poisson bracket in higher dimensions is not $-\frac{1}{\partial_z\partial_{\bar z}\ln\psi}$ but $g^{j,k}$– the inverse of the matrix $g_{j,k}$ of second derivatives for $-\ln\psi$,

$$g_{j,k} := -\partial_z^j \partial_{\bar z}^k \ln\psi.$$

Essentially, the result for correct weight remains the same, namely

$$w(z) = c_\alpha \psi^\alpha \det\left(\partial_{z_j}\partial_{\bar z_k}\ln\psi\right). \tag{9}$$

One can even generalize this approach to any Kähler manifold (see [2]).

The most difficult part is to check that Commutation relation holds

$$[T_f, T_g] = \frac{1}{\alpha} T_{\{f,g\}} + O\left(\frac{1}{\alpha^2}\right), \qquad (\alpha\to\infty), \tag{10}$$

but, remarkably, this problem can be linked (to a great extent) with the problem of establishing the asymptotic expansion of so-called *Berezin transform.* Berezin transform B_α is defined in an intrinsic way

$$B_\alpha f = \frac{\langle f K_z, K_z\rangle}{\langle K_z, K_z\rangle}.$$

The notion concerns the problem of *dequantization*, sort of inverse problem to quantization, i.e. how to assign to operators on some Hilbert space Q their symbols $\widetilde{Q}$

$$Q \to \widetilde{Q}.$$

In a Bergman space setting an example of intrinsically definable, one-to-one assignment is *Berezin symbol*,

$$\widetilde{Q}(z) = \frac{\langle Q K_z, K_z\rangle}{\langle K_z, K_z\rangle}.$$

The Berezin symbol for Toeplitz operators $Q = T_f$ is precisely Berezin transform.

Obviously, if we first quantize a function f to T_f and then dequantize it $\widetilde{T_f} = B_\alpha f$ we want to obtain the same physics, i.e.

$$B_\alpha f \to f, \qquad (\alpha\to\infty).$$

Let us assume that we have in addition a full asymptotic expansion for B_α of the special form

$$B_\alpha f \approx I + \frac{1}{4\alpha}\tilde\Delta f + \sum_{k=2}^{\infty} Q_k f, \qquad (\alpha\to\infty), \tag{11}$$

where Q_k are some linear differential operators

$$Q_k f = \sum_{\alpha,\beta \text{ multiindices}} c_{k,\alpha,\beta}\partial^\alpha \overline{\partial}^\beta f,$$

and $\tilde\Delta$ is the Laplace-Beltrami operator

$$\widetilde{\Delta} = \sum_{j,k} g^{j,k}\frac{\partial^2}{\partial z^k \partial \bar z^j}.$$

Then we can say two things. First, we can immediately construct a star product by the coefficients of the operators Q_k in the way

$$f *_{Bt} g = \sum_{k=0}^{\infty}\frac{1}{\alpha^k} C_k(f,g),$$

where

$$C_k(f,g) = \sum_{\alpha,\beta \text{ multiindices}} c_{k,\alpha,\beta} \partial^\alpha g \overline{\partial}^\beta f.$$

The fact that Q_1 is the Laplace-Beltrami operator means precisely that Commutation relation is satisfied, so it is a star product. In addition, this star product is exactly same as the star product defined via Berezin symbol, i.e. $\widetilde{Q}\widetilde{R} = \widetilde{Q *_B R}$. In other words

$$f *_{Bt} g = f *_B g.$$

And second, with some additional work we can establish

$$[T_f, T_g] \sim \frac{1}{\alpha} T_{\{f,g\}}, \qquad (\alpha \to \infty),$$

i.e. verify Commutation relation for Toeplitz operators [21, §4,2, §4,3].

For our standing example – the one dimensional Fock space $\mathcal{F}_\alpha$, expansion of this kind, indeed, exists. Actually, one can easily show that

$$B_\alpha = e^{\frac{\Delta}{4\alpha}} = \sum_{k=0}^\infty \frac{\Delta^k}{4^k \alpha^k k!}, \qquad (\alpha \to \infty).$$

For the unit disk $\mathbb{D}$ Berezin transform takes the form

$$(B_\alpha f)(z) = \int_{\mathbb{D}} f(x) \frac{(1-|z|^2)^\alpha}{|1-\bar{x}z|^{2\alpha}} (1-|x|^2)^{\alpha-2} \frac{\alpha-1}{\pi} \mathrm{d}\lambda(x).$$

This time much more work is need, but it also holds

$$B_\alpha = I + \frac{1}{4\alpha}(1-|z|^2)^2 \Delta + O\left(\alpha^{-2}\right), \qquad (\alpha \to \infty).$$

M. Engliš proved (in [21]) that the assumption we made (11) about asymptotic expansion of Berezin transform is justified for an arbitrary smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. The proof was partly based on the Feferman's theorem concerning boundary behavior of Bergman kernels on strictly pseudoconvex domains (1).

4.6. **Harmonic Berezin transform.** Bergman kernels and all notion connected to them, Berezin transform especially, exist also in the setting of *harmonic* rather than holomorphic Bergman spaces, i.e. spaces of all functions harmonic on some domain $\Omega \subset \mathbb{R}^n$ which are square integrable, with respect to a weight if necessary. The harmonic case is far from being understood. To the author knowledge only three domains was insofar studied – the whole $\mathbb{R}^n$, the unit ball $\mathbb{B}^n$ and the half-space $\mathbb{R}^n \times \mathbb{R}_+$. The continuity of point evaluation functional is assured on those domains by the mean value property.

While harmonic Berezin transform may not be directly applicable to the problem of quantization, it is still of great mathematical interested to study its asymptotic behavior and other properties.

Notably, the question of whether there is a similar asymptotic expansion as (11) is particularly appealing.

M. Engliš was able to prove for the case of harmonic Fock space $\mathbb{R}^n$ that remarkably this is the case, even though the second term of the expansion is not the Laplace-Beltrami operator and the behavior is not the same for all argument values. More precisely, he showed in 2009 [22] that the Bergman kernel takes form

$$R_\alpha(x,y) = \Phi_2\left(\begin{matrix} - \\ \frac{n}{2}-1 \end{matrix} ; \begin{matrix} \frac{n}{2}-1 & \frac{n}{2}-1 \\ - & \end{matrix} ; \alpha u_{x,y}, \alpha \bar{u}_{x,y} \right), \tag{12}$$

where $u_{x,y} = x \cdot y + \mathrm{i}\sqrt{|x|^2 |y|^2 - (x \cdot y)^2}$ and the hypergeometric function of two variables Φ_2 from Horn's list [4, §5.7.1] is defined

$$\Phi_2\left(\begin{matrix} - \\ c \end{matrix} ; \begin{matrix} b_1 & b_2 \\ - & \end{matrix} ; x, y \right) = \sum_{j,k=0}^\infty \frac{(b_1)_j (b_2)_k}{(c)_{j+k}} \frac{x^j y^k}{j! k!}.$$

In the same paper he also showed, that the corresponding Berezin transform

$$(B_\alpha f)(x) := \int_{\mathbb{R}^n} f(y) \frac{R_\alpha^2(x,y)}{R_\alpha(x,y)} \mathrm{d}\mu_\alpha^n(y),$$

where

$$\mathrm{d}\mu_\alpha^n(y) = e^{-\alpha|y|^2} \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \mathrm{d}^n y,$$

has the expansion: For $\forall f \in L^\infty(\mathbb{R}^{2n})$ which is smooth in a neighborhood of $x \neq 0$ we have

$$(B_\alpha f)(x) \approx f(x) + \frac{1}{\alpha}\left(\frac{n-2}{2}\frac{1}{|x|^2} x \cdot \nabla + \frac{(n-2)}{4(n-1)\,|x|^2}(x\cdot\nabla)^2 + \frac{1}{4(n-1)}\Delta\right) f(x) + \dots, \tag{13}$$

with additional feature that for $x = 0$ the behavior changes abruptly

$$(B_\alpha f)(0) \approx f(0) + \frac{1}{4\alpha}\Delta f(0) + \dots$$

Indeed, the terms in general asymptotic series are even singular for $x = 0$. This is an interesting occurrence of the so-called Stokes phenomenon.

For the case of the unit ball $\mathbb{B}^n$ the Bergman kernel is given by

$$R_\alpha(x,y) = F_1\left(\begin{matrix} \alpha + \frac{n}{2} + 1 \\ \frac{n}{2} - 1 \end{matrix} ; \begin{matrix} \frac{n}{2} - 1 & \frac{n}{2} - 1 \\ & - \end{matrix} ; u_{x,y}, \bar{u}_{x,y}\right),$$

where $u_{x,y}$ is as before and the first Appell function F_1 [4, §5.7.1] is defined

$$F_1\left(\begin{matrix} a \\ c \end{matrix} ; \begin{matrix} b_1 & b_2 \\ & - \end{matrix} ; x, y\right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}}{(c)_{j+k}} \frac{(b_1)_j (b_2)_k}{j!k!} x^j y^k.$$

For the case $n = 2$ the limiting behavior was confirmed to be as expected by C. Liu in 2007 [26]. That is for $f \in C(\overline{\mathbb{B}^2})$ we have,

$$B_\alpha f \to f \qquad \text{uniformly as} \quad \alpha \to \infty.$$

Subsequently, R. Otahalova in 2008 [27] generalized this result to an arbitrary dimension $n \geq 2$.

The full asymptotic expansion of the Berezin transform for the unit ball $\mathbb{B}^n$ was provided by the author in the paper [11] generalizing thus the work of Otahalova and confirming the same occurrence of the Stokes phenomenon as in the harmonic Fock space case.

More precisely, there exists a full asymptotic expansion for the harmonic Berezin transform on the unit ball $\mathbb{B}^n$ whose first two terms are

$$(B_\alpha f)(x) = f(x)$$

$$+\frac{1}{\alpha}\left(\frac{n-2}{2}\frac{1-|x|^2}{|x|^2} x \cdot \nabla + \frac{(n-2)(1-|x|^2)^2}{4(n-1)\,|x|^2}(x\cdot\nabla)^2 + \frac{1}{4(n-1)}(1-|x|^2)^2\Delta\right) f(x) + O(\alpha^{-2}), \tag{14}$$

when $x \neq 0$ and

$$(B_\alpha f)(0) \approx \sum_{i=0}^{\infty} \frac{\Delta^i f(0)}{4^i \left(\alpha + \frac{n}{2} + 1\right)_i} \qquad (\alpha \to \infty),$$

when $x = 0$.

The method used to prove this result differs substantially from methods used in [26], [27] and [22]. Notably, Otahalova's approach gives no hope to achieve this (at least as far as the present author can see), on the other hand it does not look entirely impossible to exploit the tools of the paper [22] to obtain our result but only for even dimensions.

Our approach, as we will show in the next section, is based on representing the Berezin transform in terms of generalized hypergeometric functions which asymptotic behavior is either known or can be established by more or less routine computations. Interestingly, the distinction between odd and even dimension, which burdens heavily [27] and [22], does not prove itself as important in this setting.

5. Contents of the papers [10] and [11]

5.1. **Asymptotic behavior of Bergman kernels with logarithmic weight.** The aim of [10] is to describe boundary behavior of Bergman kernel associated to the holomorphic Bergman space

$$A^2(\mathbb{D}) = \left\{ f \in \mathcal{O}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 w(z) \mathrm{d}\lambda(z) \equiv \| f \|^2 < \infty \right\},$$

with respect to the weights whose asymptotic expansions as $|z| \to 1$ are of the form:

$$w(z) \approx \left(1 - |z|^2\right)^\alpha \sum_{k=0}^{\infty} w_k \ln^{\beta-k} \frac{1}{1-|z|^2},$$

where $\alpha > -1$ and β is any real number. This is in turn achieved by first establishing the asymptotic expansion for the special case of weights

$$w(z) = \left(1 - |z|^2\right)^\alpha \left(\gamma + \ln \frac{1}{1-|z|^2}\right)^\beta,$$

where $\gamma > 0$.

Due to the radiality hypothesis, it is known that the monomials $\{z^n\}_{n \geq 0}$ form an orthogonal basis in $A^2(\mathbb{D})$, and from this it follows that Bergman kernel is given by

$$K(z, \zeta) = \sum_{n=0}^{\infty} \frac{(\overline{\zeta} z)^n}{\| z^n \|^2} \equiv K(\overline{\zeta} z).$$

The goal is to describe the behavior of $K(\overline{\zeta} z)$ as $\overline{\zeta} z \to 1$. Our main results are the following:

Theorem . *Let* $f(n) = \int_{\mathbb{D}} |z|^{2n} w(|z|^2) \mathrm{d}\lambda(z)$, *where*

$$w(t) = (1-t)^\alpha \left(\gamma + \ln \frac{1}{1-t}\right)^\beta,$$

$\alpha > -1, \gamma > 0$ *and* $\beta \in \mathbb{R}$. *The asymptotic expansion of the series*

$$K(z) = \sum_{n=0}^{\infty} \frac{z^n}{f(n)},$$

for $z \to 1$ *is:*

$$K(z) \approx \frac{\ln^{-\beta} \frac{1}{L}}{L^{\alpha+2}} \left(\alpha + 1 - \beta \ln^{-1} \frac{1}{L} + d_2 \ln^{-2} \frac{1}{L} + d_3 \ln^{-3} \frac{1}{L} + \dots \right),$$

where $L = \ln 1/z$ *and the coefficients* d_k *are given by*

$$d_k = \sum_{j=0}^{k} \binom{-\beta - j}{k - j} \Gamma^{(k-j)}(\alpha+2) c_j$$

for $c_0 = \frac{1}{\alpha!}$ *and* c_j *such that*

$$\alpha! c_j = -\sum_{i=1}^{j} \binom{\beta}{i} (\gamma - \partial_\alpha)^i \, \Gamma(\alpha+1) c_{j-i}.$$

Theorem . *The same asymptotic expansion as in the previous theorem holds for weight functions of the form:*

$$w(t) \approx (1-t)^\alpha \sum_{k=0}^{\infty} w_k \left(\gamma_k + \ln \frac{1}{1-t}\right)^{\beta-k} \qquad (t \to 1),$$

$\alpha > -1, \gamma_k > 0$ and $\beta \in \mathbb{R}$, with coefficients c_j given by $c_0 = \frac{1}{w_0 \alpha!}$ and

$$w_0 \alpha! c_j = -\sum_{i=1}^{j} \sum_{l=0}^{i} \binom{\beta - l}{i - l} w_l \left(\gamma_l - \partial_\alpha\right)^{i-l} \Gamma(\alpha+1) c_{j-i}.$$

The proof of these theorems uses the following lemma proved in the paper witch describes the asymptotic expansion of norm squares $f(n) :=\| z^n \|^2$.

Lemma . *As $n \to \infty$,*

$$\frac{n^{\alpha+1}}{\ln^\beta n} f(n) \approx \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{(-1)^k}{\ln^k n} \left(\partial_\alpha - \gamma\right)^k \Gamma(\alpha+1)$$

for $\alpha > -1, \gamma > 0$ and $\beta \in \mathbb{R}$.

Both the theorems and the lemma are proved by an ad hoc argument using just bare hands essentially following the idea behind the Laplace method. The only reference made is to an Evgrafov's book for the purpose of replacing the series representation of Bergman kernel by an appropriate integral.

Finally, at the end of the paper, we propose the following open problem. When the norm squares $f(n)$ are replaced by its principal term $\frac{\ln^\beta n}{n^{\alpha+1}}$ the corresponding approximation of Bergman kernel

$$F_{\alpha,\beta}(z) := \sum_{n=2}^{\infty} \frac{z^n n^{\alpha+1}}{\ln^\beta n},$$

extends analytically to the entire complex plane with the interval $[1, +\infty]$ removed; this is immediate from the integral representation

$$\sum_{n=k}^{\infty} \frac{z^n n^{\alpha+1}}{\ln^\beta n} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s^{\alpha+1}}{s^{[k]} \ln^\beta s} \int_1^\infty \frac{z^k k! u^s}{(u-z)^{k+1}} \, \mathrm{d}u \, \mathrm{d}s, \tag{15}$$

where k is an integer greater than $\alpha + 2$, $k - 1 < c < k$ and $s^{[k]} := s(s-1)(s-2)\dots(s-k+1)$, which is easily proved using the Residue Theorem. In particular, $F_{\alpha,\beta}$ is C^∞ on the closed unit disc except the point $z = 1$.

Unfortunately, we were unable to prove that the latter is true also for the kernel function $K(z)$, although we believe that this is the case:

Conjecture. $K \in C^\infty(\overline{\mathbb{D}} \setminus \{1\})$.

As it often happens in mathematics, the problem can be linked to a problem of seeking roots of some function. Concretely, if one can prove that the function $f(n)$ has no zeros in the right half-plane (or at least in the set $\mathrm{Re}\, n > x_0$ for some fixed x_0) a similar integral representation as (15) can be established. But at the moment the author has no idea how to approach this problem.

5.2. **Berezin transform on harmonic Bergman spaces on the real ball** [11]**.** Consider the harmonic Bergman space $L^2_{harm}(\mathbb{B}^n, \mathrm{d}\mu^n_\alpha)$ on the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, consisting of all functions that are harmonic and square integrable with respect to the measure

$$\mathrm{d}\mu^n_\alpha(y) := c_\alpha (1 - |y|^2)^\alpha \mathrm{d}^n y, \quad \alpha > -1,$$

where $\mathrm{d}^n y$ is the usual n-dimensional Lebesgue measure and the coefficient c_α is chosen so that $\mathbb{B}^n$ has measure 1. Specifically,

$$c_\alpha = \frac{\Gamma\left(\alpha + \frac{n}{2} + 1\right)}{\pi^{n/2} \Gamma(\alpha+1)}.$$

The main goal of this paper is to establish the asymptotic expansion as $\alpha \to \infty$ of the associated Berezin transform

$$(B_\alpha f)(x) := \int_{\mathbb{B}^n} f(y) \frac{R^2_\alpha(x,y)}{R_\alpha(x,x)} \mathrm{d}\mu^n_\alpha(y), \tag{16}$$

where $R_\alpha(x,y)$ is the corresponding Bergman Kernel which can be represented in terms of the Appell F_1 function

$$F_1\left(\begin{array}{c} a \\ c \end{array};\begin{array}{cc} b_1 & b_2 \\ & - \end{array};x,y\right):=\sum_{j,k=0}^{\infty}\frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}j!k!}x^jy^k,$$

as

$$R_\alpha(x,y)=F_1\left(\begin{array}{c} \alpha+\frac{n}{2}+1 \\ \frac{n}{2}-1 \end{array};\begin{array}{cc} \frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{array};u_{x,y},\bar{u}_{x,y}\right),$$

where $u_{x,y}=x\cdot y+\mathrm{i}\sqrt{|x|^2\,|y|^2-(x\cdot y)^2}$.

The main result is the following theorem.

Theorem . *For $x\in\mathbb{B}^n$, $x\neq 0$, $n>1$, and $f\in C^\infty(\mathbb{B}^n)$, there exist differential operators $Q_i:=Q_i\left(\Delta,x\cdot\nabla,|x|^2\right)$, involving only the Laplace operator Δ, the directional derivative $x\cdot\nabla$ and the quantity $|x|^2$, such that*

$$(B_\alpha f)(x):=\int_{\mathbb{B}^n}f(y)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}\mathrm{d}\mu_\alpha^n(y)\approx\sum_{i=0}^{\infty}\frac{(Q_if)(x)}{\alpha^i}\qquad(\alpha\to\infty),$$

where $Q_0=1$ and

$$Q_1=\frac{n-2}{2}\frac{1-|x|^2}{|x|^2}x\cdot\nabla+\frac{(n-2)(1-|x|^2)^2}{4(n-1)\,|x|^2}(x\cdot\nabla)^2+\frac{1}{4(n-1)}(1-|x|^2)^2\Delta.$$

Finally, for $x=0$ it holds

$$(B_\alpha f)(0)\approx\sum_{i=0}^{\infty}\frac{(\Delta^if)(0)}{4^i\left(\alpha+\frac{n}{2}+1\right)_i}\qquad(\alpha\to\infty).$$

The proof is based on representing the Berezin transform in terms of generalized hypergeometric functions

$${}_pF_q\left(\begin{array}{c} a_1\dots a_p \\ c_1\dots c_q \end{array};x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(c_1)_k\dots(c_q)_k}\frac{x^k}{k!},$$

and then make use of their many known properties including asymptotic expansions for large parameters in some cases. Concretely, we exhibit a connection between the Berezin transform of a polynomial and a linear combination of functions

$$\frac{{}_5F_4\left(\begin{array}{ccccc} \tilde{\alpha} & \tilde{\alpha} & n-2 & n-2 & \frac{n}{2}-1 \\ & \tilde{\alpha}+j_1 & \frac{n}{2}+j_2 & \frac{n}{2}+j_3 & n+j_4 \end{array};|x|^2\right)}{{}_2F_1\left(\begin{array}{cc} \tilde{\alpha} & n-2 \\ & \frac{n}{2}-1 \end{array};|x|^2\right)},\tag{17}$$

where $\tilde{\alpha}=\alpha+\frac{n}{2}+1$ and j_1,j_2,j_3,j_4 are some integers.

Unfortunately, the needed asymptotic expansion of ${}_5F_4$ for large parameters was at the time not to be found in the literature (to the best of the authors knowledge). Hence, we gave in the paper the proof of the following lemma.

Lemma . *Let $b_1,b_2,b_3>0$ be positive real numbers, one of them strictly less than the other two. Let $\alpha-a-\gamma\notin\mathbb{Z}$, $-c_i\notin\mathbb{N}_0$ and $x\in(0,1)$. Then we have*

$${}_5F_4\left(\begin{array}{ccccc} \alpha & \alpha & b_1 & b_2 & b_3 \\ & \alpha+a & c_1 & c_2 & c_3 \end{array};x\right)\approx\prod_{i=1}^{3}\frac{\Gamma(c_i)}{\Gamma(b_i)}\frac{(\alpha x)^{-\gamma}}{(1-x)^{\alpha-\gamma-a}}\left(1+\sum_{k=1}^{\infty}\frac{d_k}{\alpha^k}\right)\qquad(\alpha\to+\infty),$$

where $\gamma=\sum_{j=1}^{3}(c_j-b_j)$ and d_k are constants independent of α.

The lemma includes as a special case the asymptotic expansion of Gauss hypergeometric function ${}_2F_1$ in the denominator of (17) which is also needed, though its expansion is well known.

Finally, along the way we proved the following theorem which bears some significance of its own, since it provides means of computing the Bergman projection of more general functions than just harmonic ones:

Theorem . *For* $\forall p \in \mathbb{N}_0$, $\beta \geq \alpha$ *and* $f \in C^p(\mathbb{B}^n) : \Delta f = 0$ *it holds*:

$$\int\limits_{\mathbb{B}^n} R_\alpha(x,y) f(y) (x\cdot y)^p \mathrm{d}\mu^n_\beta(y)$$

$$= \frac{p!}{2^p} \sum_{j+2l+m=p} \frac{|x|^{2(j+l)} (\tilde{\alpha})_j (2b)_j}{j!m!l!(\tilde{\beta})_{j+m+l}(b)_j} \; {}_3\left((x\cdot\nabla)^m f\right)_3 \left(\begin{matrix} \tilde{\alpha}+j & 2b+j & b \\ \tilde{\beta}+j+l+m & b+j & 2b \end{matrix} ; x \right),$$

where $b := \frac{n}{2} - 1$ *and* $\tilde{x} := x + \frac{n}{2} + 1$.

Note that in the case $\beta = \alpha$ and $p = 0$ this reads

$$\int\limits_{\mathbb{B}^n} R_\alpha(x,y) f(y) \mathrm{d}\mu^n_\alpha(y) = f(x),$$

thus we recover the reproducing property.

Here we have introduced the "hypergeometrization" ${}_mf_n$ of a function f, which is a special case of a Hadamard product and which appears naturally in this setting.

More precisely, for a real (or complex) function f of a real argument we define its *hypergeometrization* by the series

$${}_pf_q \left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; t \right) := \sum_{m=0}^{\infty} \frac{t^m f^{(m)}(0)}{m!} \frac{(a_1)_m \dots (a_p)_m}{(c_1)_m \dots (c_q)_m},$$

whenever this defines some analytic function in a neiqhbourhood of zero – i.e. the radius of convergence R is strictly greater than zero and none of the lower parameters c_i is a non-positive integer.

And for a real function $f(x)$ of a vector argument, $x \in \mathbb{R}^n$, $n > 1$ we define

$${}_pf_q \left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; x \right) := {}_pf_q \left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; tx \right) \Bigg|_{t=1},$$

that is the hypergeometrization is performed on the real function $f(tx)$ of the real argument t and if the corresponding radius of convergence is strictly grater than 1 then the function is evaluated at the point $t = 1$.

6. Presentations related to the thesis

(1) 21th International Workshop on Operator Theory and Applications, IWOTA, July 2010, Berlin, Germany.
Talk: The asymptotic behavior of Bergman kernels with logarithmic weight
(2) Göttingen 2011: Summer School. Analysis – with Applications to Mathematical Physics, 29. 8.–2. 9. Germany.
Talk: The asymptotic behavior of Bergman kernels with logarithmic weight.
(3) International Conference on Differential Equations, Difference Equations and Special Functions, 3.9-7.9, 2012, Patras, Greece.
Talk: Harmonic Bergman projection formula.

7. Publications constituting the body of the thesis

[1] P. Blaschke: *Asymptotic behavior of Bergman kernels with logarithmic weight.* J. Math. Anal. Appl. 385 (2012), no. 1, 293–302.
[2] P. Blaschke: *Berezin transform on harmonic Bergman spaces on the real ball*, J. Math. Anal. Appl., 2013, DOI:10.1016/j.jmaa.2013.09.056

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