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1. Introduction

The thesis is based on four independent papers [P1] – [P4]. The common subject is the theory of discrete dynamical systems generated by continuous maps of a compact interval and a circle into itself.

The first part (Section 3) of the thesis provides a “universal” continuous function of the interval. This solves a problem formulated by A. M. Bruckner (cf., e.g., [Br]). The second part (Section 4) gives a characterization of transitive points for continuous transitive maps of the interval. The third part (Section 5) extends some recent results from the class of continuous maps of the interval to the class of continuous maps of the circle. First we provide a “universal” continuous function of the circle and give a characterization of transitive points for continuous transitive maps of the circle. At the end of the section we give a geometric characterization of ω-limit sets and show that the family of ω-limit sets is closed with respect to the Hausdorff metric.

2. Basic terminology and notation

Throughout this abstract the set of continuous maps from a compact metric space Y into itself will be denoted by $\mathcal{C}(Y,Y)$. Symbols I and $\mathbb{S}$ denote the unit interval $[0,1]$ and the circle $\{z \in \mathbb{C}; |z| = 1\}$, respectively. By X we denote either the interval I or the circle $\mathbb{S}$. Recall that the *trajectory* of a point x under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the n-th iteration of f. If there is $k \geq 1$ such that $f^k(x) = x$ and $f^n(x) \neq x$ for every $n = 1, \ldots, k-1$ then x is a *periodic* point with the *period* k. The set of limit points of the trajectory of x is called *ω-limit set* and we denote the set by $\omega_f(x)$. By ω_f we denote the system of ω-limit sets $\omega_f(x)$ where $x \in X$. The map f is *transitive* if for every two non-empty open sets V, W there is a positive integer n such, that $f^n(V) \cap W \neq \emptyset$. The point $x \in X$ is called *transitive* point of the map f if the point x has a dense trajectory in X. The set of transitive points of f is denoted by $\mathrm{Tr}(f)$.

Denote by $e : \mathbb{R} \to \mathbb{S}$ the natural projection defined by $e(x) = \exp(2\pi \mathrm{i} x)$. Note that the map $\tilde{e} : (0,1) \to \mathbb{S} \setminus \{e(0)\}$ obtained by restricting e to the interval $(0,1)$, is a homeomorphism. We say that $\tilde{e}(x) \leq \tilde{e}(y)$ whenever $x \leq y$. For an interval $A \subset \mathbb{S} \setminus \{e(0)\}$ a point a is called the *left endpoint*, resp. the *right endpoint*, of A if $a \leq x$, resp. $x \leq a$, for every $x \in A$. We say that a set $A \subset \mathbb{S}$ is *T-side* or *T-unilateral neighborhood* (T means either “left” or “right”) of an $x \in \mathbb{S}$ if the set A is a closed interval and the point x is T endpoint of the set A.

Let $U = U_0 \cup \ldots \cup U_{N-1}$ be the union of pairwise disjoint non-degenerate closed intervals and $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$. For any set $K \subset U$ let $f_U(K) = f(K) \cap U$ (this may be empty). Inductively define $f_U^n(K) = f_U(f_U^{n-1}(K))$. Let $\tilde{K} \equiv \tilde{K}(U) = \bigcup_{i=1}^{\infty} f_U^i(K)$;

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although $\tilde{K}$ depends on U, to avoid convoluted notation we use $\tilde{K}$ whenever the set U is evident. Let $A \subset \mathbb{S}$ be a closed set and $x \in A$. We say that a side T of a point x is *A-covering* if for any union of finitely many closed intervals U such that $A \subset \mathrm{Int}(U)$ and any closed T-unilateral neighborhood $V(x)$ there are finitely many components of $\tilde{V}(x)$ such that the closure of their union covers A. If T is an A-covering side of x then any T-unilateral neighborhood $V(x)$ is also said to be *A-covering*. If every $x \in A$ has an A-covering side we call the set A *locally expanding* according to the map f.

A *Cantor set* is any nowhere dense non-empty compact set without isolated points. A *homeomorphic copy* of a set $A \subset X$ is a copy with respect to an order-preserving homeomorphism. In this abstract by interval we mean a non-degenerate one, exceptions are stated explicitely.

For more terminology see standard books like [Al] or [Bc].

3. "Universal" dynamical system

The following characterization of ω-limit sets is due to Agronsky et al. [Ag]: A non-empty compact set $F \subset I$ is an ω-limit set of a map $f \in \mathcal{C}(I, I)$ if and only if F is either a finite collection of compact intervals, or nowhere dense. A characterization of sets in ω_f, for any fixed continuous f, is given in [Bk]. The system ω_f equipped with the Hausdorff metric, is a compact set [Bk].

In view of the above mentioned facts there is the following natural problem: how large can the system ω_f be? In [K] Keller gives a simple example of a function $f : I \to I$ continuous everywhere except for a single point such that any nowhere dense compact set $F \subset I$ has a homeomorphic copy $\tilde{F}$ in ω_f; the corresponding homeomorphism can be extended to the whole of I. By Evans et al. [E], if $f \in \mathcal{C}(I, I)$ has periodic trajectories of all periods then any non-empty countable compact set has a homeomorphic copy in ω_f. This homeomorphism, however, cannot be in general extended from F to the whole interval I. In this section we state a theorem that there is a "universal" continuous function, up to homeomorphisms of the interval, solving a problem formulated by A. M. Bruckner (cf., e.g., [Br]).

Theorem A. *There is a map $f \in \mathcal{C}(I, I)$ such that, for any non-empty compact set $F \subset I$ which is either nowhere dense or is the union of finitely many intervals, there is a homeomorhism φ from I to I such that $\varphi(F)$ is an ω-limit set of f.*

Clearly, the condition "up to homeomorphism" cannot be omitted, and the function involved cannot be simple. In our case f is strongly irregular, having infinite variation on any open interval which intersects a Cantor set C_f. Moreover, the ω-limit sets of f which are contained in C_f, form our universal system for infinite nowhere dense ω-limit sets.

4. Structure of sets of transitive points

It is well-known that for a continuous map f of a compact metric space Y, the set $\mathrm{Tr}(f)$ of transitive points of f, i.e. the set of points in Y with dense trajectory, is a dense G_δ set. The argument is straightforward: Denote by $\mathcal{B}$ a countable base of Y. The set of transitive points of the map f is then of the form $\mathrm{Tr}(f) = \bigcap_{G\in\mathcal{B}} \bigcup_{n=1}^{\infty} f^{-n}(G)$ and therefore, a G_δ set, which is dense since any $\bigcup_{n=1}^{\infty} f^{-n}(G)$ is dense.

But, not every dense, G_δ set is a set of transitive points of a continuous map. In this section we give a characterization of sets of transitive points, solving a problem formulated by Ľ. Snoha.

Theorem B. *A set $T \subset I$ is a set of transitive points for a continuous map $f : I \to I$ if and only if $I \setminus T$ is a first category set, F_σ and c-dense in I.*

In paper [P3] we give two different proofs of the theorem. The first proof is based on two Sharkovsky's results (see [S]) and is not trivial. The second proof is more elementary.

5. Dynamical systems on the circle

Since a compact interval and a circle are both one-dimensional connected compact sets, continuous maps of the interval and continuous maps of the circle have many properties in common. However to transfer results from the interval to the circle, we usually have to make some modifications that do not have to be seen at once. A good example is the classical result of the continuous maps of the interval, the Sharkovsky's theorem (see [Bc]). The theorem does not hold for the maps of the circle in this classical version and it has to be modified for this class of maps (see [Bc]).

Recall that X denotes either the compact unit interval or the unit circle $\mathbb{S}$.

Note that in Section 3 we mentioned a known fact (see [Ag]): a non-empty compact set $F \subset I$ is an ω-limit set of a map $f \in \mathcal{C}(I, I)$ if and only if F is either a finite collection of compact intervals, or nowhere dense. The following theorem extends this fact to $\mathcal{C}(X, X)$.

Theorem C. *A non empty compact set $F \subset X$ is an ω-limit set of a map $f \in \mathcal{C}(X, X)$ if and only if F is either a finite collection of compact intervals or a nowhere dense set.*

From Section 3 we have the "universal" function of the interval. In this section we construct a "universal" continuous function of the circle. Denote the "universal" function of the interval I by h. Consider a function ψ by shrinking the function h from $[0, 1]$ to $[1/3, 2/3]$ and extended it linearly to the whole circle so that ψ is continuous and $\psi(0) = 0$ and $\psi(1) = 1$. To finish the construction denote $g = e \circ \psi \circ (e|_{[0,1)})^{-1}$ where e is the natural projection. Corresponding theorem is the following one.

Theorem D. *There is a map $g \in \mathcal{C}(X, X)$ such that for any $f \in \mathcal{C}(X, X)$ and any ω-limit set $\omega_f(x) \neq X$, there is a homeomorphism φ from X into X and a point $y \in X$ such that $\varphi(\omega_f(x)) = \omega_g(y)$.*

The theorem cannot be further improved by extending the function to cover even the set $F = X$. The case $F = \mathbb{S}$ is not possible, because the only homeomorphic copy is F and a function $g \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ possessing this ω-limit set cannot have any other ω-limit set.

Section 4 gives us a characterization of the sets of transitive points for continuous maps of the interval. Extension of the theorem to the class $\mathcal{C}(X, X)$ is given in the following theorem.

Theorem E. *A set $T \subset X$ is a set of transitive points for a map $f \in \mathcal{C}(X, X)$ if and only if $X \setminus T$ is a first category, F_σ set c-dense in X or $T = X$ and $X = \mathbb{S}$.*

In [Bk] Blokh, Bruckner, Humke and Smítal gave geometric characterization of ω-limit sets of maps in $\mathcal{C}(I, I)$ and proved that the family of ω-limit sets of a map in $\mathcal{C}(I, I)$ is closed with respect to the Hausdorff metric. We extend these results to the class $\mathcal{C}(X, X)$.

Theorem F. *Let f be a map in $\mathcal{C}(X, X)$. Then the family of all ω-limit sets of f endowed with the Hausdorff metric is compact.*

The following theorem is a criterion for a set to be an ω-limit set.

Theorem G. *Let f be a map in $\mathcal{C}(X, X)$. A closed set $A \subset X$ is an ω-limit set if and only if it is locally expanding.*

The next theorem is a stronger generalization of old Sharkovsky's results (see [S]).

Theorem H. *Let $\{\omega_n\}_{n=1}^\infty = \{\omega_f(x_n)\}_{n=1}^\infty$ be a sequence of ω-limit sets of a continuous map $f \in \mathcal{C}(X, X)$ and let a point a has a side T, such that for any T-unilateral neighborhood V of a, there exists a positive integer N such that for each $n \geq N$, the trajectory of x_n enters V infinitely many times. Then $\bigcap_{k=1}^\infty \overline{\bigcup_{n=k}^\infty \omega_n}$ is an ω-limit set.*

6. Publications concerning the thesis

[1] D. Pokluda, J. Smítal, *An omega-limit set universal function on* $[0, 1]$, Real Analysis Exchange, **24** (1998/1999), 109 – 110. Abstract of [3].

[2] D. Pokluda, *On the structure of sets of transitive points for continuous maps of the interval*, Real Analysis Exchange, **25** (1999/2000), 45 – 48.

[3] D. Pokluda, J. Smítal, *A "universal" dynamical system generated by a continuous map of the interval*, Proc. Amer. Math. Soc. **128** (2000), 3047 – 3056.

[4] D. Pokluda, *On the transitive and ω-limit points of the continuous mappings of the circle*, Preprint Series in Mathematical Analysis 20/2000, Silesian University, Opava. Accepted to Archivum Mathematicum.

[5] D. Pokluda, *Characterization of ω-limit sets of continuous maps of the circle*, Preprint Series in Mathematical Analysis 23/2001, Silesian University, Opava.

7. Quotations by other authors

[6] M. Málek, *ω-limit sets for continuous circle maps*, Preprint Series in Mathematical Analysis, Silesian University, Opava. (cf. [5])

8. Presentations

[7] 1st Czech - Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, May 31 – June 4, 1997. Talk on: "A universal continuous function."

[8] 26th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January, 23 – 29, 1998. Talk on: "A universal continuous function."

[9] 2nd Czech - Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, May 7 – 10, 1998. Talk on: "An omega-limit set of universal function on $[0, 1]$."

[10] European Conference on Iteration Theory – ECIT 98, Muszyna, Poland, August 30 – September 5, 1998. Invitation. Talk on: "A universal dynamical system generated by a continuous map of the interval."

[11] 23th Summer Symposium in Real Analysis, Łódź, Poland, June 21 – 26, 1999. Talk on: "On the structure of transitive points."

[12] 3rd Czech - Slovak Conference on Dynamical Systems, Liptovský Trnovec, Slovakia, September 23 – 29, 1999. Talk on: "On the structure of transitive points."

[13] 28th Winter School in Abstract Analysis, Křišťanovice, Czech Republic, January 23 – 29, 2000.

[14] 4th Czech - Slovak Conference on Dynamical Systems, Praděd, Czech Republic, June 22 – 28, 2000. Talk on: "The set of points with a dense orbit."

[15] 29th Winter School in Abstract Analysis, Lhota nad Rohanovem, February 3 – 10, 2001. Talk on: "Characterization of ω-limit sets of continuous maps of the circle."

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[K] P. S. Keller, *Chaotic behavior of Newton's method*, Real Anal. Exchange **18 (1992/93)**, 490 – 507.

[P1] D. Pokluda, J. Smítal, *A "universal" dynamical system generated by a continuous map of the interval*, Proc. Amer. Math. Soc. **128** (2000), 3047 – 3056.

[P2] D. Pokluda, *On the structure of sets of transitive points for continuous maps of the interval*, Real Analysis Exchange, **25** (1999/2000), 45 – 48.

[P3] D. Pokluda, *On the transitive and ω-limit points of the continuous mappings of the circle*, Preprint Series in Mathematical Analysis 20/2000, Silesian University, Opava. Accepted to Archivum Mathematicum.

[P4] D. Pokluda, *Characterization of ω-limit sets of continuous maps of the circle*, Preprint Series in Mathematical Analysis 23/2001, Silesian University, Opava.

[S] A. N. Sharkovsky, *The partially ordered sets of attracting sets*, Soviet Math. Dokl **7 (1966)**, 1384 – 1386.