Silesian University in Opava
Mathematical Institute

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Distributional spectrum of dynamical systems

Abstract of the Ph.D. Thesis
January 2021

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav

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Distribuční spectrum dynamických systémů

Autoreferát dizertační práce
leden 2021

Matematická analýza

Výsledky tvořící dizertační práci byly získány během magisterského a doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2015–2021.

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Školící pracoviště:	Matematický ústav SU, Opava

Oponenti:

Autoreferát byl rozeslán dne:

Státní doktorská zkouška a obhajoba dizertační práce se konají dne 19.2.2021 v hod. v zasedací místnosti rektorátu Slezské univerzity v Opavě.

S disertací je možno se seznámit v knihovně Matematického ústavu SU, Na Rybníčku 1, Opava.

Předseda oborové rady: prof. RNDr. Miroslav Engliš, DrSc.
Matematický ústav SU, Opava

CONTENTS

1. INTRODUCTION

The thesis concerns two papers, both of them deal with the spectrum and the weak spectrum of distributional functions in two separeted cases. In their paper [8], Schweizer and Smítal have defined the notion of distributional chaos for continuous map of interval, the spectrum and the weak spectrum of a dynamical system. Among other things, they have proved that in the case of continuous interval maps, both the spectrum and the weak spectrum are finite and nonempty. The spectrum and weak spectrum were also studied in paper [5].

In the first paper we generalized result from [8] to a continuous map on a tree. The second paper is devoted to the spectrum and the weak spectrum of piecewise monotonic map of interval. Both papers have two parts. The first part contains the main theorem and its proof. In the second part, we present examples to show that spectrum will be infinite if we weaken conditions of our systems.

2. BASIC TERMINOLOGY AND NOTATIONS

Throughout this thesis assume that X is a compact metric space with the metric d and f is map from X into X, together they from a *Dynamical system* (X, f). Recall that the *trajectory* of a point $x \in X$ under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the *n*-th iteration of f. A point $x \in X$ is called a *periodic point* if there is a $k \geq 1$ such that $f^k(x) = x$, smallest such k is called the *period* of x. The ω*-limit set* of a point $x \in X$ is the set of accumulation points of trajectory of x and we denote it as $\omega_f(x)$. We say that $\omega_f(x)$ is *maximal* if there is no $y \in X$ such that $\omega_f(x) \subsetneq \omega_f(y)$.

Based on work of A. N. Sharkovsky in [6, 7], maximal ω-limit set were characterised and their properties widely studied [1, 2]. If ω-limit set is finite then it is a cycle. If it is infinite and does not contain a periodic point then is called a solenoid. The ω-limit set is called a basic set if it is infinite and contains a periodic point. The last case of ω-limit sets plays crucial role in distributional chaos. In the case of continuous map on tree and also in the case of piecewise monotonic map on interval with Markov condition and generator we have three types of ω-limit set described above.

An *arc* is a topological space homeomorphic to the compact interval I. A *graph* is a nonempty compact connected metric space (a continuum) which can be written as the union of finitely many arcs and two of which can intersect only in their endpoints. If a graph does not contain a set homeomorphic to the circle, it is called a *tree*. A *dendroid* is an arcwise

The research was funded by institutional support for the development of research organization (IČ47813059) and by SGS 18/2019

connected, hereditarily unicoherent continuum (a tree-like continuum).

Let I be a compact interval, a map $f : I \to I$ is called *piecewise monotone* if there is a finite colection $\mathcal{J}$ of pairwise disjoint nondegenerate intervals $\{J_1, \ldots, J_N\}$, such that $f|_{\mathrm{int}(J_i)}$ is continuous and strictly monotone, and f is semicontinuous at $c \in \mathrm{fr}(J_i)$. Endpoints of intervals J_i are called critical points. Union of intervals from $\mathcal{J}$ is equal to I.
We suppose that our piecewise monotonic map satisfies *Markov condition* and has *generator*. We say that piecewise monotonic map f satisfies *Markov condition* if for any critical point c one-sided limits are in the set of critical points. The collection of intervals $\mathcal{J}$ is *the generator* if for every sequence $\{k_i\}_{i=0}^{\infty}$ of integers $k_i \in \{1, 2, \ldots, N\}$ the set $\bigcap_{j=0}^{\infty} f^{-j}(J_{k_j})$ contains at most one point. Properties of piecewise monotonic maps were studied by Hofbauer and Raith [3, 4].

We continue with definitions of the spectrum and the weak spectrum as were defined by Schweizer and Smítal in [8, 9]. Let for $x, y \in X$ and for a positive integer n be $\delta_{xy}(n) = \mathrm{d}(f^n(x), f^n(y))$ the distance of *n*-th iteration of points x, y. Then for any $x, y \in X$, a positive integer n and any real t, let

$$\xi(x, y, t, n) = \#\{i; 0 \leq i \leq n-1 \text{ and } \delta_{xy}(i) < t\} \tag{1}$$

where $\#S$ is the cardinality of the set S. Put

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n), \tag{2}$$

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n). \tag{3}$$

Function F^*_{xy} is called an *upper distributional function* and F_{xy} is called *lower distributional function*. Both F^*_{xy} and $F_{x,y}$ are nondecreasing functions with $F^*_{xy}(t) = F_{xy}(t) = 0$ for every $t < 0$ and $F^*_{xy}(t) = F_{xy}(t) = 1$ for every $t > \mathrm{diam}(X)$, where $\mathrm{diam}(X)$ is diameter of X.

Let f be a map on space X and $x, y \in X$. Then the pair (x, y) is *isotectic* (with respect to f) if, for every positive integer n, the ω-limit sets $\omega_{f^n}(x)$ and $\omega_{f^n}(y)$ are subsets of the same maximal ω-limit set of f^n.

Equivalently, the pair (x, y) is isotectic if ω-limit sets $\omega_f(x)$ and $\omega_f(y)$ are subset of the same maximal ω-limit set $\tilde{\omega}$ and if for any f-periodic set J such that $J \cup f(J) \cup f^2(J) \cup \cdots \cup f^{m-1}(J) \supset \tilde{\omega}$, where m is the period of J, there is $j \geq 0$ such that both $f^j(x)$ and $f^j(y)$ belong to J. Note that this equivavalent definition was introduced in [8] and such points are called *synchronous*.

The *spectrum of* f, denoted by $\Sigma(f)$, is a set of minimal elements of the set $D(f) = \{F_{xy} |\, x \text{ and } y \text{ are isotetic}\}$ and the *weak spectrum of* f, denoted by $\Sigma_w(f)$, is a set of minimal elements of the set $D_w(f) = \{F_{xy} |\, \liminf_{n\to\infty} \delta_{xy}(n) = 0\}$.

3. Spectrum on tree

In the first part of the first paper we deal with spectrum of distributional functions of a continuous self map on tree.

Theorem A. *Let f be a continuous self map of a tree.*

(1) *If the topological entropy of f is zero, then $\Sigma(f) = \{\chi_{(0,\infty)}\}$.*
(2) *If the topological entropy of f is positive, then the both spectrum $\Sigma(f)$ and the weak spectrum $\Sigma_w(f)$ are finite and nonempty. Specifically, $\Sigma(f) = \{F_1, \dots, F_m\}$ for some $m \geq 1$, and $\Sigma_w(f) \setminus \Sigma(f) = \{F_{m+1}, \dots, F_n\}$ for some $n \geq m$. Futhermore, for each i there is an $\epsilon_i > 0$, such that $F_i(\epsilon_i) = 0$.*

In the second part of this paper we present example of continuous self map of a dendroid, where the spectrum and weak spectrum are both infinite.

4. Spectrum of piecewise monotonic map

The second paper deals with the case of piecewise monotonic map on interval. The discontinuity of a map brought us many challenges, since we were not able to use classic tools for investigation of a continuous map. In the first part of this paper we show that under certain conditions the spectrum of piecewise monotonic map is always finite.

Theorem B. *Let f be a piecewise monotonic map with Markov condition and generator, then both the Spectrum $\Sigma(f)$ and the Weak Spectrum $\Sigma_w(f)$ are nonempty and finite.*

We were able to show that our system has positive topological entropy

Corollary C. *Any piecewise monotonic map f with Markov condition and generator has an iteration with horseshoe and consequently f has positive topological entropy.*

In the case of piecewise monotonic map f with generator $\mathcal{J}$, which does not satysfies Markov condition, we can change the collection $\mathcal{J}$ to $\mathcal{J}'$ such that $\mathcal{J}'$ is also generator and f satysfies Markov condition, but $\mathcal{J}'$ can have an infinite number of members.

In the second part we present example of piecewise monotonic map, which has no generator and infinite spectrum.

5. Publications concerning the thesis

[1] J. Tesarčík, *On the spectrum of dynamical system on trees*, Topology and its Applications 222 (2017), 227–237

[2] V. Pravec, J. Tesarčík, *On distributional spectrum of piecewise monotonic maps*, (submitted to Ergodic Theory and Dynamical Systems in December 2020)

6. Presentations and participations

[3] The sixth Visegrad conference on dynamical systems, Praha, Czech Republic, July, 6–10, 2015. Talk on: "On Spectrum of dynamical system on tree. "

[4] Czech-Slovak workshop on discrete Dynamical systems, Karlova Studánka, Czech Republic, September, 12–16, 2016.

[5] Czech, Slovenian, Austrian, Slovak and Catalan mathematical societies joint meeting, Barcelona, Spain, September, 20–23, 2016. Talk on: "On the Spectrum of dynamical system on tree. "

[6] 7th Visegrad conference on dynamical systems, Opava, Czech Republic, June, 26–30, 2017.

[7] 23rd International conference on difference equations and applications (ICDEA), Timisoara, Romania, July, 24–28, 2017. Talk on: " On the spectrum of dynamical system on tree."

[8] Wandering seminar, A mini-course on Ergodic theory and dynamical system, Łódź, Poland, December, 7–10, 2017.

[9] Czech-Slovak workshop on dynamical systems, Banská Bystrica, Slovakia, June, 19–22, 2018.

[10] International workshop and conference on Topology and Applications (IWCTA), Cochin, Kerala, India, December, 5–8 & 9–11, 2018. Talk on: " Some properties of piecewise monotonic maps with markov condition."

[11] 8th Visegrad conference on dynamical system, Budapest, Hungary, June, 24–28, 2019. Talk on: "On Spectrum of piecewise monotonic

maps with markov condition."

[12] Dynamics, Geometry and Analysis: 20 zears of Mathematical Institute in Opava, Hradec nad Moravicí, Czech Republic, September, 8–13, 2019. Talk on: "On Spectrum of piecewise monotonic maps with markov condition."

[13] 17th International conference of numerical analysis and applied mathematics (ICNAAM), Rhodes, Greece, September, 23–28, 2019. Talk on: "On Spectrum of piecewise monotonic maps with markov condition."

7. Long-term visits

[14] University of Vienna, 1.March – 31.May, 2017.

[15] University of Vienna, 1.October 2018 – 31.March, 2019.

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[9] B. Schweizer, A. Sklar, J. Smítal *Distributional (and other) chaos and its measurement*, Real Analysis Exchange 26(2) (2000), 495–524