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Opinion on the dissertation
Conjugacy equation in dimension one
and its applications in iteration theory
by Krzysztof Ciepliński

The dissertation under review has been composed of six papers (three due to the author alone, and three more written jointly with M. C. Zdun). A self-report has been joined, which is a concise presentation of main results contained in the papers. A large and comprehensive list of references has been added. The topic of the thesis proposed by the candidate for promotion is conjugacy of mappings. This is an important matter in the theory of dynamical systems, or iteration theory. Below I am analyzing the content of papers.

1. The most important seems the paper quoted by the Author as [25], written jointly with M. C. Zdun and published in *Functional equations - results and advances,* Kluwer Acad. Publ., Dordrecht 2002, 135–158.

The paper large both in volume and in touched problems, contains a quite complete description of the following problem. Given two families $\mathcal{F} = \{F_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 : t \in M\}$ and $\mathcal{G} = \{G_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 : t \in M\}$, where M is an arbitrary non-empty set, of pairwise commuting homeomorphisms of the unit circle $\mathbb{S}^1$, is it possible to find a continuous (homeomorphic) solution $\Phi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of the system

$$\Phi \circ F_t = G_t \circ \Phi, \ t \in M? \tag{0.1}$$

The authors answer this question formulating a series of theorems. Some of them (Theorem 3.1.1 = Theorem 1 in [25]) generalize and correct some earlier results (in this particular case a result form [6] by I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai). They are also extending some earlier results (Proposition 3.1.1 = Proposition 1 in [25] is claimed to extend a result by H. Poincaré, although it is not said which of results of one of the most prominent mathematicians).

Roughly speaking, the existence and uniqueness of solutions of (0.1) is strictly connected to the structure of limit sets and rationality (or not) of rotation number(s) of F_t and G_t. For instance, the Theorem 3.1.3 (= Theorem 3 in [25]), under assumptions of pairwise commutativity of $\mathcal{F}$ and $\mathcal{G}$, and equality (mod1) of rotation numbers of G_t and F_t (the last one multiplied by some integer ℓ), and, additionally, existence of an index t_0 such that the rotation number of G_{t_0} is irrational, links the existence and uniqueness of solutions of (0.1) to the structure of the limit set for G_{t_0} andF_{t_0}. All these results are very interesting, certainly original and bringing new ideas to the iteration theory. The proofs are correct, and in some cases very non-standard, showing the intellectual power of the authors who very ingeniously yield arguments. Also the remarks are putting light on the considered problems. The example concluding the paper is very nice and underlining the nuances of Theorem 3.1.4 (= Theorem 4 in [25]).

2. Strictly connected with [25] is the paper [26], dealing with a system of Schröder equations on the circle, and written also jointly with M. C. Zdun. The paper has been published in 2003, in the *International Journal of Bifurcation and Chaos* (vol. 13, pp. 1883-1888). The paper deals with a specialized problem of semiconjugacy, namely the authors consider the system of equations

$$\Phi \circ F_t = c(t)\Phi, \ t \in M, \tag{0.2}$$

where assumptions on $\mathcal{F}$ are relaxed - this is now a family of pairwise commuting *continuous* functions mapping $\mathbb{S}^1$ into itself.

In the case of card$M$ = 1, Theorem 3.2.1 (= Theorem 2.1 in [26]) has been proved which states that if a continuous mapping $\Phi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfies

$$\Phi(f(z)) = s\Phi(z), \ z \in \mathbb{S}^1,$$

and deg$\Phi \neq 0$ then there exists a finite, and independent of x or f, limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n}, \ x \in \mathbb{R}$$

(here $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an arbitrary *lift* of F). We see that the above limit is a natural generalization of rotation number for F, earlier defined for orientation preserving homeomorphisms. Moreover then deg$F = 1$ and

$$s = \exp 2\pi i\alpha(F)\deg\Phi.$$

The authors note that if deg$\Phi = 0$ then $s = 1$. In the case of arbitrary M the authors got Theorem 3.2.2 (= Theorem A in [26]). The result is connecting the existence of a unique pair (Φ, c) of solutions of (0.2) with the behaviour of the set

$$C(z) := \{(F_{t_1}^{n_1} \circ \cdots \circ F_{t_k}^{n_k})(z) : t_1, \ldots, t_k \in M, n_1, \ldots, n_k \in \mathbb{Z}, k \in \mathbb{N}\}, \ z \in \mathbb{S}^1.$$

There are many Corollaries in the paper. For instance, if we additionally assume that $\mathcal{F}$ is composed of pairwise commuting homeomorphisms without fixed points then (Corollary 3.2.1 = Corollary 3.1 in [26]) there always exists a(Φ, c) satisfying (0.2). Moreover, if the set $\{(F_{t_1}^{n_1} \circ \cdots \circ F_{t_k}^{n_k})(z) : t_1, \ldots, t_k \in M, n_1, \ldots, n_k \in \mathbb{Z}, k \in \mathbb{N}\}$ is infinite for every $z \in \mathbb{S}^1$ then Φ has to be continuous. In my opinion it is a very interesting result, reflecting the main interest behind functional equations: to get regularity (almost) uniquely from the equation itself. On the other hand, we find also results of the type "solutions depend on an arbitrary function" in [26] (it concerns the case where F has some periodic points or $\alpha(F) \in \mathbb{Q}$). Also in this paper we find the phrase "result of Poincaré" - this time, while still not quoted with details, it is at least mentioned as "well known", so, in my opinion, it sounds a little bit less mysterious.

3. The third paper, *Topological conjugacy of disjoint flows on the circle* is due solely to Krzysztof Ciepliński, and has been published in Bull. Korean Math. Soc. 39 (2002), 333–346. This time the author considers the case where $M = V$ is an Abelian group with cardV > 1 and $\mathcal{F}$ is an iteration group, i.e. the translation equation is satisfied:

$$F^{v_1+v_2} = F^{v_1} \circ F^{v_2}.$$

Again, the question is when two iteration groups, or *flows*, are conjugated, i. e. there exists a homeomorphism Φ such that the equation (0.1) is satisfied. The author assumes that the flows are disjoint and is interested in the case where mappings are defined (and taking values) in $\mathbb{S}^1$ - earlier M. C. Zdun studied the case of flows defined in an open real interval. There are three cases, depending on the limit set $L_{\mathcal{F}} := \{F^v : v \in V\}$ is equal to $\mathbb{S}^1$ (then the flow is called dense),or $\emptyset \neq L_{\mathcal{F}} \neq \mathbb{S}^1$ (non-dense flow), or $L_{\mathcal{F}} = \emptyset$ (discrete flow). In any case, sufficient and necessary conditions are given, which guarantee the topological conjugacy of two flows. The results are correct and very interesting.

One result that is specially interesting to me is Proposition 3.3.2 (= Proposition 3 in [16] or in [18]). The author in his self-report writes the following: *Let us* *mention here that Theorem* 3.2.2 *plays a crucial role in the proof of this proposition (see [18]).* It seems to me that he means in fact Theorem 3.3.1 from the self-report. However, it is hard to check because the author did not include the paper [18], *The structure of disjoint iteration groups on the circle*, published in Czechoslovak Math. J. 54 (2004), 131–153, to the dissertation. Therefore I will not comment on this particular result, in which two interesting functional equations are explicitly appearing.

4. The fourth paper of six composing the dissertation, is entitled *On conditions guaranteeing that mappings are elements of iteration groups.* It was written by Dr. Ciepliński and was published in Applied Mathematical Letters 24 (2011), 1415–1418. It brings yet another answer to a question of J. Schwaiger about functions commuting with all members of an iteration group: do they necessarily belong to the group? The author answers positively to the question in the case of a group of mappings of an open interval I.

Actually, it turns out that if the group is *strictly disjoint* (i.e. if $f^v \in \mathcal{F}$ has a fixed point then $v = 0$) then either $L_{\mathcal{F}} = \mathrm{cl} I$ or $L_{\mathcal{F}}$ is a perfect nowhere dense subset of I. In the first case we say that $\mathcal{F}$ is *dense.* The author quotes a Lemma 3.4.2 (= Lemma 5 in [23]) stating that in this case $\mathcal{F}$ is of the form

$$f^v(x) = \varphi^{-1}(\varphi(x) + c(v)), \ x \in I, \ v \in V,$$

where $\varphi : I \longrightarrow \mathbb{R}$ is an increasing homeomorphism, and $c : V \longrightarrow \mathbb{R}$ is an additive injection. Using the Lemma and a result of J. Matkowski, the author is able to prove that if $\mathcal{F}$ is a dense flow and $g : I \longrightarrow I$ is continuous at least at one point and commutes with two arbitrarily chosen mappings f^a, $f^b \in \mathcal{F}$ such that $\frac{a}{b} \neq \mathbb{Q}$ then g is topologically conjugate to a translation. Moreover, if $\mathcal{F}$ is *non-singular* (i.e. the function c from Lemma 3.4.2 is bijective), then $g \in \mathcal{F}$.

The author extends this result to the case where g satisfies two inequalities (instead of equalities), with assuming additionally that $f^a < \mathrm{id} < f^b$ (Theorem 3.4.2 = Theorem 2 in [23]). This is also a very nice result. Similarly two other results contained in the paper (Theorem 3.4.3 = Theorem 3 in [23], and Corollary 3.4.2 = Corollary 2 in [23]) present two interesting statements concerning two flows, one of the non-singular and the other continuous (i.e. such that for every $x \in I$ the mapping $\mathbb{R} \ni t \longrightarrow g^t(x) \in I$ is continuous). It turns out that the equality

$$g^v \circ f^v = f^v \circ g^v, \ v \in \mathbb{R},$$

implies the existence of an additive function $C : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g^v = f^{C(v)}$, or, in particular,

$$g^v \circ f^u = f^u \circ g^v, \ v, u \in \mathbb{R}.$$

5. The investigation of problems connected with mappings commuting with flows are continued in the paper *Schröder equation and commuting functions on the circle*, written by Dr. Ciepliński and published in J. Math. Anal. Appl. 342 (2008), 394–397. This time he deals with flows on the unit circle $\mathbb{S}^1$. It starts with Theorem 3.5.1 (= Theorem 1 in [22]), stating that if $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is an orientation preserving homeomorphism without periodic points then the equation

$$\Phi(F(z)) = \exp 2\pi i \alpha(F)\Phi(z), \ z \in \mathbb{S}^1,$$

has a unique (up to a multiplicative constant) solution $\Phi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ continuous at a point of $L_F = \{F^n(z) : n \in \mathbb{Z}\}^d$. Moreover, this solution is continuous.

This is again an extension to a "well known result of Poincaré". The main results read as follows.

a) Suppose that for a $c \in \mathbb{R} \setminus \{0\}$ we have $F^t(z) = \Phi\left(\exp(2\pi ict\Phi^{-1}(z)\right)$, for $t \in \mathbb{R}$ or $t \in (0, \infty)$, where $\Phi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism. If a function $G : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is continuous at least at one point, commutes with with a mapping F^a and $ca \notin \mathbb{Q}$ then $G = F^t$ for some $t \in J \in \{\mathbb{R}, (0, \infty)\}$.
b) Under the assumptions of a), if $\mathcal{G}$ is a continuous iteration semigroup of continuous at least at one point mappings G^t such that
$$F^t \circ G^t = G^t \circ F^t,$$
for every $t > 0$ then there exists a function $C : (0, \infty) \longrightarrow J$ such that
$$C(s + t) - C(s) - C(t) \in \frac{\mathbb{Z}}{c}, \; s, t > 0,$$
and $G^t = F^{C(t)}$, $t > 0$.

When we compare the results for open interval and the circle, we note that they are of a little bit different nature. First, it is enough to assume that G commutes with only one mapping (however, a special one) and on the other hand, the function C is now a congruence. It opens opportunities for somebody willing to obtain further results from classical ones concerning functional congruences. Anyway, I am sure that these results are very valuable.

6. The last paper constituting the dissertation is again a joint paper with M. C. Zdun, entitled *On uniqueness of conjugacy of continuous and piecewise monotone functions.* It was published in Fixed Point Theory and Applications (2009), ID 230414, 11 pages, and is of a different character.

First of all it is now concentrated on mappings of closed real intervals into itself. They are now supposed to be *piecewise monotone* and continuously mapping each piece onto the domain. Such mappings are known in the literature as *horseshoe maps*. The authors consider two horseshoe maps $f : I \longrightarrow I$ and $g : J \longrightarrow J$ of the same type (roughly speaking with the same number of subintervals (called *laps*) mapped by them monotonically on the whole interval) and ask for solutions of conjugacy equation

$$\varphi \circ f = g \circ \varphi \tag{0.3}$$

where $\varphi : I \longrightarrow J$. They prove (Theorem 3.6.1 = Theorem 3.1 in [27]) that if f, g are of the same type (with n laps) and additionally g is γ-expansive then there exists a unique function φ satisfying (0.3) and
$$\varphi(I_i) \subset J_i, \; i \in \{0, \ldots, n-1\}.$$
The authors prove moreover that the function φ is continuous, surjective and increasing, and if f is piecewise expansive, then φ is strictly increasing.

The result is very nice and the method of proof (fixed point theory) is interesting. The other two theorems in the paper (Theorem 3.6.2 = Theorem 3.2 in [27] and Theorem 3.6.3 = Theorem 3.5 in [27]) are of a similar character.

Let me summarize my remarks and present a conclusion.

It is not clear for me why some papers were omitted, even though if, in my opinion, they deserve to be included in the dissertation. I have to say that the papers from which the dissertation has been composed are written in a routine way. There are not too many new ideas. However, those appearing in the dissertation are interesting and presented often in a clever and ingenious manner. The notation is sometimes very difficult to follow, similarly as a great number of definitions. I have had some problems with following the argument in two or three papers. For instance, I have been long looking for the reference (6) in the paper [16] (this is the number of a functional equation which has been interesting to me) - finally I have been successful and found the reference (p. 344, l. 15 from above).

Nevertheless I have to say that the results are new and interesting, in a positive meaning of the words.

The conjugacy equation is of a great importance not only in iteration theory but also in the theory of dynamical systems, and more generally, in mathematics. The quality of presented results is highly satisfactory.

Moreover, the results are promising further developments and opening new areas for future investigation. For instance a natural question arises: what in the case of iteration semigroups? The subject is touched only in one paper. I understand that the treatment rather of groups than semigroups depends highly on the tools the author(s) is (are) using.

I am convinced that from the scholar point of view the dissertation presented by Dr. Krzysztof Ciepliński satisfies the usual requirements for habilitation and I do support his claim for promotion.