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Harmonic Bergman spaces and related problems

Abstract of the Ph.D. Thesis
July 2015

Mathematical Analysis

Slezská univerzita v Opavě
Matematický ústav v Opavě

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Harmonické Bergmanovy prostory a související problémy

Autoreferát dizertační práce
Červenec 2015

Matematická analýza

Výsledky uvedené v této dizertační práci byly získány v průběhu mého doktorského studia oboru Matematická analýza na Matematickém ústavu Slezské univerzity v Opavě v letech 2010–2015. Výzkum byl podporován projekty SGS/17/2010 a SGS/1/2013.

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Autoreferát této dizertační práce byl zveřejněn dne 15. července 2015.

Státní doktorská zkouška a obhajoba dizertační práce se budou konat dne 19. listopadu 2015 v 11:30 hodin před oborovou komisí doktorského studia Matematická analýza v budově Slezské univerzity v Opavě, Na Rybníčku 1, Opava.

Dizertační práce je dostupná v knihovně Matematického ústavu, Na Rybníčku 1, Opava.

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The author would like to thank prof. RNDr. Miroslav Engliš, DrSc. for his invaluable help and support during the research and work on this thesis.

Contents

1. Introduction

This thesis is based on two independent papers [22] and [12], the common subject of which are harmonic and holomorphic Bergman spaces with reproducing kernels (the so called Bergman kernels) and the asymptotic behaviour of the associated Berezin transform of one or two arguments. Both these papers constitute an integral part of the thesis.

The theory of reproducing kernels, of which the theory of Bergman spaces is a part, has its roots already at the beginning of the twentieth century in the work of J. Mercer, E. H. Moore and S. Zaremba, see [3] for a nice overview of the state-of-the-art towards 1950. It was however S. Bergman who introduced what he called "kernel functions" in his thesis [10] in 1922 for the first time as kernels corresponding to classes of harmonic and analytic functions in one or several variables with the reproducing property (Bergman kernels). A great deal of important results were achieved using these kernels ranging from the theory of functions of one and several complex variables to classification of biholomorphic strongly pseudoconvex domains to the theory of invariant metrics (the Bergman metric) to partial differential equations, see for example [9], [26], [18].

From the functional analytic and operator theoretic point of view, the Bergman spaces, whose precise definition is given in Section 2, can be viewed as a natural outgrowth of the theory of Hardy spaces H^p, $0 < p < \infty$, which in case $p = 2$ is closely connected with the analogous theory of Szegö kernels, see [6] for a nice unified treatment of Bergman and Szegö kernels via the so called Kerzman-Stein formula, and [14], [34] for general information on functional analytic aspects of the theory of Bergman spaces.

Yet another area, where the theory of Bergman spaces finds its extensive use is the part of mathematical physics dealing with the so called quantization procedures. Here the explicit knowledge of the relevant reproducing kernels and their asymptotic behaviour together with some related concepts derived from them (the Berezin symbols, the Toeplitz operators and the Berezin transform) play significant role[1], which we try to illustrate in Section 4, where the basic idea of the so called Berezin quantization due to F. A. Berezin [8] is briefly described. For an allied idea of Berezin-Toeplitz quantization as well as some other quantization procedures we refer to the survey paper [1] and references therein. A short review of basic concepts from classical and quantum mechanics that serves as a way to introduce the readers into a broader context in which quantization takes place is given in Section 3. We emphasize the fact that this rather motivational section, heavily biased to physics, does not exactly fit the expertise area of the author and is only indirectly related to the main theme of the thesis.

It is quite an interesting fact that some of the results pertaining to quantization schemes on domains in $\mathbb{C}^n$ using the *holomorphic* Bergman spaces which, after making necessary

[1]At least within the class of analytic functions on certain domains in $\mathbb{C}^n$.

technical adjustments, work also on general Kähler manifolds, in fact remain in force also for more general *harmonic* function spaces on certain open subsets of $\mathbb{R}^n$ (the harmonic Bergman spaces) even if their applicability to quantization seems to be virtually irrelevant. In Section 5 we provide two theorems of this sort that are due to Engliš [16] and Blaschke [11].

Finally, the two aforementioned papers [22] and [12] are given some attention in Sections 6 and 7, respectively.

The first paper [22] entitled *"On asymptotic expansion of the harmonic Berezin transform on the half-space"* was published in *Journal of Mathematical Analysis and Applications* in 2013 and its main result is a theorem on asymptotic expansion of the harmonic Berezin transform, analogous to the expansions cited in Section 5, this time for the harmonic Bergman space on the half-space in $\mathbb{R}^n$.

The second paper [12] with the title *"Berezin transform of two arguments"*, published in *Journal of Functional Analysis* in 2015 aims at a deeper conceptual insight into the structure of the Berezin transform by extending it to an integral transform of two arguments and showing that, in some special cases, similar expansions to the expansions mentioned above do also hold true even in this setting.

We would like to remark that, with the exception of the ideas presented in the papers [22] and [12], all results in this abstract have probably already appeared elsewhere and in this sense we make no claims to originality.

2. Holomorphic and harmonic Bergman spaces

In this section, we briefly review some basic facts related to certain reproducing kernel Hilbert spaces, namely the so called holomorphic and harmonic Bergman spaces that play a key role in subsequent considerations. The main purpose is to fix notation for later reference. A basic source of the material covered here are the books [25] and [4].

2.1. **The holomorphic case.** Let $\Omega \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$, i.e. a connected open subset in $\mathbb{C}^n$. Denote by $\mathcal{O}(\Omega)$ the space of functions that are holomorphic on Ω and consider the Lebesgue volume measure λ on $\mathbb{C}^n$ and the corresponding space $L^2(\Omega, \lambda)$ of square-integrable functions with respect to λ. We define the **holomorphic Bergman space** on Ω to be the set

$$A^2(\Omega) = \{f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2 \,\mathrm{d}\lambda(z) < \infty\} = \mathcal{O}(\Omega) \cap L^2(\Omega, \lambda). \tag{1}$$

Using the mean-value property for holomorphic functions it is easily shown that for every compact set $K \subset \Omega$ there is a constant $C = C(K, n) > 0$ such that for every $f \in A^2(\Omega)$

$$\sup_{z \in K} |f(z)| \leq C \|f\|_{A^2(\Omega)}. \tag{2}$$

Two important consequences of this last result are the following: first of all, the estimate (2) implies that the convergence in norm implies the uniform convergence on compact subsets of Ω, which eventually implies that the limit of an arbitrary Cauchy sequence of functions in $A^2(\Omega)$ (the limit exists since L^2 is complete) is in fact a holomorphic function. This is tantamount to the fact that $A^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(f,g) = \int_\Omega f(z)\overline{g(z)}\,\mathrm{d}\lambda(z). \tag{3}$$

Second, taking the set K to be a one-point set $\{z\}$, where z is a fixed point in Ω, and the mapping e_z (the so called **evaluation functional**) whose action on functions is defined by

$$A^2(\Omega) \ni f \mapsto f(z) \in \mathbb{C},$$

we can see that, due to (2), e_z is a bounded linear functional on the Hilbert space $A^2(\Omega)$. Therefore, invoking Riesz representation theorem, there is a function $K_z \in A^2(\Omega)$ such that for every $f \in A^2(\Omega)$ and for every $z \in \Omega$

$$\begin{aligned} e_z(f) &= f(z) = (f, K_z) \\ &= \int_\Omega f(x)\overline{K_z(x)}\,\mathrm{d}\lambda(x). \end{aligned}$$

Denoting $K(x,z) := K_z(x)$ we can treat the complex function K as a mapping from $\Omega \times \Omega$ and we have

$$\begin{aligned} K(x,z) &= K_z(x) \\ &= (K_z, K_x) && \text{(due to the reproducing property of } K\text{)} \\ &= \overline{(K_x, K_z)} && \text{(since the product (3) is sesquilinear)} \\ &= \overline{K_x(z)} && \text{(the reproducing property once again)} \\ &= \overline{K(z,x)}. \end{aligned}$$

This also means that not only the reproducing kernel $K(\cdot,\cdot)$ is holomorphic in the first variable (as $K(\cdot,z)$ belongs to $A^2(\Omega)$ for every $z \in \Omega$) but it is antiholomorphic in the second variable.

Next we recall a result that may be sometimes useful in practical computations:

Theorem 1 (see [25])**.** *Let $\{\varphi_j\}$ be a any complete orthonormal system for $A^2(\Omega)$. Then the series*

$$\sum_{j=0}^{\infty} \varphi_j(x)\overline{\varphi_j(z)}$$

sums to the Bergman kernel $K(x,z)$, uniformly on $C \times C$ for every compact set $C \subset \Omega$.

Although to compute the Bergman kernel in an explicit form for a given domain Ω is usually quite a formidable task, in certain special instances it is nevertheless feasible. An example, known essentialy already to Poincaré, of a domain for which this is indeed possible and the mother of them all, is the following (see [24]):

Example 1. Set $\Omega = \mathbb{D}$, the unit disc in $\mathbb{C}$. We first note that for every $j \neq k$

$$(z^j, z^k) = \int_{\mathbb{D}} z^j \overline{z}^k \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^{2\pi} r^{j+k} \mathrm{e}^{\mathrm{i}\theta(j-k)} r \, \mathrm{d}\theta \, \mathrm{d}r = 0.$$

Since

$$\left(\int_{\mathbb{D}} |z^j|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} = \left(\int_0^{2\pi} \int_0^1 r^{2j+1} \, \mathrm{d}r \, \mathrm{d}\theta \right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{\sqrt{j+1}},$$

it is seen that the system $\left\{ \frac{\sqrt{j+1}}{\sqrt{\pi}} z^j \right\}$ is orthonormal. Moreover, if we write f as a Taylor series, $f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=0}^{\infty} a_k r^k \mathrm{e}^{\mathrm{i}k\theta}$, we have for $0 < R < 1$ by uniform convergence

$$\begin{aligned}
&\int_{D(0,R)} \frac{\sqrt{j+1}}{\sqrt{\pi}} z^j \overline{f} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^R \left(\int_0^{2\pi} \frac{r^j \mathrm{e}^{\mathrm{i}j\theta} \sqrt{j+1}}{\sqrt{\pi}} \sum_{k=0}^{\infty} a_k r^k \mathrm{e}^{-\mathrm{i}k\theta} r \, \mathrm{d}\theta \right) \mathrm{d}r \\
&= 2\pi \left(\frac{a_j \sqrt{j+1}}{\sqrt{\pi}} \int_0^R r^{2j+1} \, \mathrm{d}r \right).
\end{aligned}$$

Clearly,

$$\lim_{R \to 1^-} \int_{D(0,R)} \frac{\sqrt{j+1}}{\sqrt{\pi}} z^j \overline{f} \, \mathrm{d}x \, \mathrm{d}y = \left(\frac{\sqrt{j+1}}{\sqrt{\pi}} z^j, f \right),$$

so that the assumption $\left(\frac{\sqrt{j+1}}{\sqrt{\pi}} z^j, f \right) = 0$ for every j implies that $a_j = 0$ for every j, hence $f \equiv 0$. Thus the system $\left\{ \frac{\sqrt{j+1}}{\sqrt{\pi}} z^j \right\}$ is also complete. It follows from Theorem 1 that

$$K(x, z) = \sum_{j=0}^{\infty} \frac{\sqrt{j+1}}{\sqrt{\pi}} x^j \frac{\sqrt{j+1}}{\sqrt{\pi}} \overline{z}^j = \frac{1}{\pi} \sum_{j=0}^{\infty} (j+1)(x\overline{z})^j.$$

The last sum is just a differentiated geometric series so that finally

$$K(x, z) = \frac{1}{\pi} \frac{1}{(1 - x\overline{z})^2},$$

which is what we wanted to show. ■

Remark 1. The method that we have used to compute the Bergman kernel on $\mathbb{D}$ in Example 1 is not the only one existing. For an alternative approach using Green's functions see [6].

For reasons that will become shortly apparent, it is also important to study the so-called weighted analogues of the classical Bergman spaces. Suppose ρ is a positive Lebesgue measurable real-valued function on Ω. Then we call ρ **a weight function** on Ω and the space $L^2(\Omega, \rho)$ is the so called $\rho-$**weighted L^2 space,** which means that $f \in L^2(\Omega, \rho)$ if and only if

$$\int_{\Omega} |f(z)|^2 \rho(z) \, \mathrm{d}\lambda(z) < \infty.$$

Modifying the standard L^2- case in a self-evident manner, it is seen that $L^2(\Omega, \rho)$ is a Hilbert space with the inner product

$$(f, g) = \int_\Omega f(z)\overline{g(z)}\rho(z)\,\mathrm{d}\lambda(z).$$

Moreover, everything that was done above in the unweighted case goes through unchanged under some relatively mild additional conditions on ρ. Namely: if ρ is such a weight function, for which $1/\rho$ is locally integrable[2] on Ω, then it can be shown that the corresponding $\rho-$weighted variant of the holomorphic Bergman space (1), defined by

$$A^2(\Omega, \rho) = \{f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2\rho(z)\,\mathrm{d}\lambda(z) < \infty\} = \mathcal{O}(\Omega) \cap L^2(\Omega, \rho), \tag{4}$$

is in fact a closed linear subspace of $L^2(\Omega, \rho)$, hence a Hilbert (sub)space (of $L^2(\Omega, \rho\lambda)$) in its own right, sometimes possibly trivial (depending on a particular choice of ρ), for which the evaluation functional e_z is continuous for every fixed z just as in the unweighted case. The corresponding reproducing kernel is called **the weighted Bergman kernel** and is usually denoted by K_ρ.

Important special instances of the weighted Bergman spaces and their respective Bergman kernels are the following:

Example 2. $\Omega = \mathbb{D}$, $\rho(z) = \frac{\alpha+1}{\pi}(1 - |z|^2)^\alpha$. It can be shown (see for example [34]) that the factor $\frac{\alpha+1}{\pi}$ in $\rho(z)$ is chosen in such a way that

$$\frac{\alpha+1}{\pi}\int_\Omega (1 - |z|^2)^\alpha\,\mathrm{d}\lambda(z) = 1$$

for every $\alpha > -1$. Similarly to Example 1 it can be shown that the corresponding weighted kernel K_ρ is given by the formula

$$K_\rho(x, y) = \frac{1}{(1 - x\overline{y})^{\alpha+2}},$$

which, up to the factor $1/\pi$, reduces to $K(x, y)$ from Example 1 if $\alpha = 0$ (the unweighted case). ■

Example 3. $\Omega =$ the unit ball in $\mathbb{C}^n$, $\rho(z) = c_\alpha(1 - \|z\|^2)^\alpha$, where again the coefficient c_α (which depends on α) is chosen in such a way to make $\rho(z)$ of total mass one, to be precise

$$c_\alpha = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)\pi^n}.$$

The corresponding weighted Bergman kernel is

$$K_\rho = \frac{1}{(1 - \langle x, y\rangle)^{\alpha+n+1}},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard hermitian inner product in $\mathbb{C}^n$, see [30]. ■

[2] This is usually unnecessarily general condition on ρ; in virtually every place in this thesis, positivity and continuity of the function ρ will do.

In what follows, we shall frequently take $\alpha = 1/h$, where h plays the role of Planck's constant (see Sections 3.5 and 3.6) if not explicitly stated otherwise. If we want to stress the dependence of the weights ρ on the parameter α or h, we shall write ρ_α or ρ_h, respectively (and similarly for the corresponding function spaces).

Example 4. The space $A^2(\Omega, \rho)$ (sometimes denoted also by $\mathfrak{F}_h$), where $\Omega = \mathbb{C}^n$ (i.e. the functions in $\mathfrak{F}_h$ are entire on $\mathbb{C}^n$) and with the weight function $\rho(z) = \mathrm{e}^{-\alpha|z|^2}(\alpha/\pi)^n$. This is the so called **Segal-Bargmann** or **Fock** space. The reproducing kernel is given by the formula

$$K_\rho(x, y) = \mathrm{e}^{\langle x,y\rangle/h} = \mathrm{e}^{\alpha\langle x,y\rangle},$$

see [35]. ■

2.2. **The harmonic case.** Consider again the space $L^2(\Omega, \lambda)$ of functions that are square-integrable on a nonempty open subset $\Omega \subset \mathbb{R}^n$ with respect to the usual Lebesgue measure λ on Ω. In a fashion completely parallel to (1) and (4), we can define the so called **harmonic Bergman space** $B^2(\Omega)$ to be the set

$$B^2(\Omega) = \{f \in \mathcal{A}(\Omega) : \int_\Omega |f(z)|^2 \,\mathrm{d}\lambda(z) < \infty\} = \mathcal{A}(\Omega) \cap L^2(\Omega, \lambda),$$

where $\mathcal{A}(\Omega)$ is the space of all harmonic functions on Ω. Since, as already tacitly pointed out in subsection 2.1, the weighted Bergman spaces have essentially nothing special, and we can thus subsume both the $\rho-$weighted and unweighted Bergman spaces under one heading by considering the (harmonic or holomorphic) Bergman spaces as subspaces of $L^2(\Omega, \mu)$ with respect to an appropriate measure μ on Ω such that $\rho = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$, where ρ is a positive continuous[3] function and $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is the Radon-Nikodym derivative (i.e. the measure μ has a positive continuous density with respect to the Lebesgue measure). Under these conditions it can be shown that, using the mean value property for harmonic (or holomorphic) functions, the evaluation functional e_x is continuous for every fixed $x \in \Omega$, similarly to the holomorphic case. The corresponding reproducing kernel $H_x(\cdot) \in B^2(\Omega, \rho)$ (we shall write $B^2(\Omega, \rho)$ whenever we want to stress the dependence of the space $B^2(\Omega)$ on the weight ρ) which is obviously a harmonic function for every $x \in \Omega$, is called **the (weighted) harmonic Bergman kernel** with the **reproducing property:**

$$f(x) = (f, H_x) = \int_\Omega f(y)\overline{H_x(y)} \,\mathrm{d}\mu(y).$$

Denoting $H(x, y) := H_y(x)$, the harmonic Bergman kernel can be viewed as a function on $\Omega \times \Omega$ and due to the reproducing property we have

$$H(x, y) = H_y(x) = (H_y, H_x) = \overline{(H_x, H_y)} = \overline{H_x(y)} = \overline{H(y, x)} \tag{5}$$

[3]These conditions are again not of the utmost generality, but they are sufficient for our purposes.

and, for every $f \in B^2(\Omega,\rho)$, we thus obtain, using the fact that if f is a harmonic function, then $\overline{f}$ is a harmonic function, that

$$\begin{aligned}\overline{(f,H_y)} &= \overline{f(y)} = (\overline{f},H_y) = \int_\Omega \overline{f(z)H_y(z)}\,\mathrm{d}\mu(z)\\ &= \overline{\int_\Omega f(z)\overline{\overline{H_y(z)}}\,\mathrm{d}\mu(z)} = \overline{(f,\overline{H_y})},\end{aligned}$$

showing that $H_y = \overline{H_y}$, which implies that, unlike in the holomorphic case, the function $H(x,y)$ is in fact a real-valued function and that (using (5)) $H(x,y) = H(y,x)$.

Remark 2. Both the definitions of the holomorphic and harmonic Bergman spaces can be extended to the realm of L^p spaces for $1 \le p < \infty$ and their weighted variants. However, we shall make no use of this stuff in the sequel, see [34].

A handful of examples of (weighted) harmonic Bergman spaces and associated Bergman kernels is known. They are essentially the following:

Example 5. For $\Omega = \mathbf{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$, the open upper half-space in $\mathbb{R}^n$ with the unweighted Lebesgue measure, the reproducing kernel for the space $B^2(\Omega)$ si given by

$$K(x,y) = \frac{2\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\frac{(n-1)(x_n+y_n)^2+(x_n-y_n)^2-|x-y|^2}{((x_n+y_n)^2-(x_n-y_n)^2+|x-y|^2)^{\frac{n}{2}+1}},$$

see e.g. [4]. ■

Example 6 (see [16])**.** The space $\mathcal{F}_h$ from Example 4 has the following space $\mathcal{H}_h$ as its harmonic counterpart: we take $\Omega = \mathbb{R}^n$ and the weight function $\rho(x)$ is much the same as in Example 4:

$$\rho(x) = \left(\frac{1}{\pi h}\right)^{\frac{n}{2}} \mathrm{e}^{-\frac{\|x\|^2}{h}}.$$

The harmonic Bergman kernel for $B^2(\Omega,\rho) = \mathcal{H}_h$ can be computed explicitly to be given for $n \ge 3$ by the formula

$$H_\rho(x,y) = \Phi_2\left(\begin{array}{c}-\\ \frac{n}{2}-1\end{array};\begin{array}{cc}\frac{n}{2}-1 & \frac{n}{2}-1\\ & -\end{array};\frac{t_1+\mathrm{i}t_2}{h},\frac{t_1-\mathrm{i}t_2}{h}\right), \tag{6}$$

where

$$t_1 = \langle x,y\rangle, \quad t_2 = \sqrt{|x|^2|y|^2-\langle x,y\rangle^2}$$

and Φ_2 is one of the hypergeometric functions of two variables from Horn's list, defined as

$$\Phi_2\left(\begin{array}{c}-\\ c\end{array};\begin{array}{cc}a & b\\ & -\end{array};z,w\right) = \sum_{j,k=0}^{\infty}\frac{(a)_j(b)_k}{(c)_{j+k}j!k!}z^jw^j, \tag{7}$$

where $(a)_k := a(a+1)(a+2)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the so called Pochhammer symbol defined whenever a and $a+k$ are not negative inetgers (we remark that the series for Φ_2 converges for every z, $w \in \mathbb{C}$ and thus defines an entire function on $\mathbb{C}^2$, see [5] or [31].) For

$n = 2$, the harmonic Bergman kernel is twice the real part of the usual holomorphic Bergman kernel on $\mathbb{C}$ minus one, for $n = 1$

$$H_\rho(x, y) = 1 + \frac{2xy}{h}, \quad x, y \in \mathbb{R}.$$

The details of the proof of the formula (6) as well as the discussion of the remaining cases are to be found in [16]. ■

Example 7. For $\Omega = \mathbb{B}^n$, the unit ball in $\mathbb{R}^n$ with the weight function $\rho(x) = c_\alpha(1 - |y|^2)^\alpha$, where the coefficient c_α is given by

$$c_\alpha = \frac{\Gamma\left(\alpha + \frac{n}{2} + 1\right)}{\pi^{n/2}\Gamma(\alpha + 1)},$$

it has been computed in many places that for $\alpha > -1$ the reproducing kernel K_ρ for the space $B^2(\Omega, \rho)$ is

$$K_\rho(x, y) = F_1\left(\begin{matrix} \alpha + \frac{n}{2} + 1 \\ \frac{n}{2} - 1 \end{matrix}\, ; \begin{matrix} \frac{n}{2} - 1 & \frac{n}{2} - 1 \\ & - \end{matrix}\, ; z, \overline{z}\right), \tag{8}$$

where

$$F_1\left(\begin{matrix} a \\ c \end{matrix}\, ; \begin{matrix} b_1 & b_2 \\ & - \end{matrix}\, ; x, y\right) = \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}j!k!} x^j y^k \tag{9}$$

is one of the hypergeometric functions of Appell and $z = x \cdot y + \mathrm{i}\sqrt{|x|^2|y|^2 - (x \cdot y)^2}$ (here the series on the right of (9) converges for every $x, y \in \mathbb{C}$ with $\max\{|x|, |y|\} < 1$, see [31].) For the details on (8) see [11]. ■

3. Classical versus quantum mechanics

In this section we give a short account of basic principles of classical and quantum mechanics in their Hamiltonian/Heisenberg formulation, which later serves as a motivation behind some of the quantization schemes introduced in later (sub)sections. For the sake of brevity, the whole discussion is essentially limited to the case of a particle moving in $\mathbb{R}^n$, i.e. the corresponding phase space M is just $\mathbb{R}^{2n}$ if not stated otherwise. The treatment closely follows [19] and [33]. Would like once more to stress the motivational character of this section and to point out that its relevance to the main theme of the thesis is only indirect.

3.1. **Classical mechanics.** Consider the problem of a particle moving in $\mathbb{R}^n$ under the action of a force $\mathbf{F}$. Suppose the position $\mathbf{x} = \mathbf{x}(x_1, \ldots, x_n)$ of the particle is a function of its canonical coordinates in $\mathbb{R}^n$ (which may of course depend on time t). It is well-known that for a conservative force field $\mathbf{F}$, i.e. such that there exists a function $V(\mathbf{x})$ for which

$$\mathbf{F} = -\operatorname{grad} V \equiv -\frac{\partial V}{\partial \mathbf{x}},$$

Newton's second law reads

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\operatorname{grad} V, \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v},$$

or, using the change of variables $\mathbf{p} = m\mathbf{v}$,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial V}{\partial \mathbf{x}}, \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathbf{p}}{m}. \tag{10}$$

If we note that for the function $H(\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$, there are the relations

$$\frac{\mathbf{p}}{m} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{\partial V}{\partial \mathbf{x}} = \frac{\partial H}{\partial \mathbf{x}},$$

then the system (10) can be obviously recast in the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{11}$$

Thus conservative mechanical systems are a special incarnation of the so called **Hamiltonian systems** which in fact comprise much broader classes of ordinary differential equations of the form

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{12}$$

where $H(\mathbf{q},\mathbf{p}) = H(q_1,\dots,q_n,p_1,\dots,p_n)$ is the so called **Hamiltonian function,** q_i and p_i are the so called **generalized coordinates** and **generalized momenta,** respectively, and n stands for the **number of degrees of freedom.**

3.2. **The structure of the algebra of observables.** One of the deepest facts connecting classical physics with mathematics is a kind of duality between points of a manifold M and functions on that manifold, which says that points $x \in M$ can be identified with $\mathbb{R}-$homomorphisms $\mathscr{F} \to \mathbb{R}$, where $\mathscr{F}$ is an appropriate $\mathbb{R}-$algebra of smooth functions on the manifold. If we interpret the manifold M as a model of a given physical system and $\mathscr{F}$ as a collection of measuring devices and the points of M as the states of the system, then it should not be surprising that any classical physical system is described by an appropriate algebra of smooth functions on the phase space M, each state $x \in M$ of the system being the homomorphism $x : \mathscr{F} \to \mathbb{R}$ that the state x determines on $\mathscr{F}$, see [27] for a deeper discussion of this issue. In case $M = \mathbb{R}^{2n}$, denote by $\mathscr{F}$ the commutative algebra of smooth real-valued functions on M (we recall M is the so called **phase space** alluded to above, whose coordinates are denoted by $q_1,\dots,q_n,p_1,\dots,p_n$) and call the functions $f \in \mathscr{F}$ the **(classical) observables.** In this sense, the Hamiltonian functions are observables and the Cauchy problem for a Hamiltonian system (12):

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1,\dots,n$$

with initial conditions $q_i\big|_{t=0} = q_0^i$, $p_i\big|_{t=0} = p_0^i$, $i = 1,\dots,n$ and the Hamiltonian function $H = H(q_1,\dots,q_n,p_1,\dots,p_n)$, has a unique local solution of the form

$$q_i = q_i(q_0^i, p_0^i, t), \quad p_i = p_i(q_0^i, p_0^i, t)$$

(here and in what follows we assume that the solutions are actually defined on the whole of $\mathbb{R}$).

The solutions to the Hamiltonian system described above define a **one-parameter group of transformations** G_t of $\mathbb{R}^{2n}$ into itself:

$$G_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n},$$

where $G_t(\mathbf{q}, \mathbf{p})$ is *the* solution of the Hamiltonian system corresponding to the initial condition $G_t(\mathbf{q}, \mathbf{p})\big|_{t=0} = (\mathbf{q}, \mathbf{p})$ and it follows that

$$G_{t+s} = G_t G_s = G_s G_t \quad \text{and} \quad G_t^{-1} = G_{-t}.$$

On the level of the algebra of observables $\mathscr{F}$, the transformations G_t have the effect that they generate a system of transformations $U_t : \mathscr{F} \to \mathscr{F}$ of the algebra of observables given by

$$U_t f(\mathbf{q}, \mathbf{p}) := f(G_t(\mathbf{q}, \mathbf{p})) \equiv f_t(\mathbf{q}, \mathbf{p}).$$

We remark that the values of the function f_t are completely determined by the initial conditions $\mathbf{q_0}, \mathbf{p_0}$. This fact is reflected in the coordinate description of $f_t(\mathbf{q}, \mathbf{p})$ in the form

$$f_t(\mathbf{q_0}, \mathbf{p_0}) = f(\mathbf{q}(\mathbf{q_0}, \mathbf{p_0}, t), \mathbf{p}(\mathbf{q_0}, \mathbf{p_0}, t)). \tag{13}$$

It is now a standard fact that the function $f_t(\mathbf{q}, \mathbf{p})$ satisfies the differential equation

$$\frac{\partial f_t}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial f_t}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f_t}{\partial p_i} \frac{\partial H}{\partial q_i} \right),$$

with the initial condition $f_t(\mathbf{q}, \mathbf{p})\big|_{t=0} = f(\mathbf{q}, \mathbf{p})$ or, in much compact form using (13),

$$\frac{\mathrm{d} f_t}{\mathrm{d} t} = \{H, f_t\}, \tag{14}$$

where we have defined

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right),$$

the so called **(canonical) Poisson bracket,** for arbitrary observables f, g. The following algebraic properties of the Poisson bracket can be established in a straightforward manner:

(1) $\{f, g\} = -\{g, f\}$; (antisymmetry)
(2) $\{f, g + ch\} = \{f, g\} + c\{f, h\}$; (linearity)
(3) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$; (Jacobi identity)
(4) $\{f, gh\} = g\{f, h\} + \{f, g\}h$. (product rule)

In particular, according to the product rule, the Poisson bracket is a derivation on the algebra $\mathscr{F}$ in the sense that

$$X_f(gh) = (X_f g)h + g(X_f h),$$

where X_f is a vector field on $\mathbb{R}^{2n}$ defined by

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} \right).$$

To sum up, the algebra $\mathscr{F}$ is a commutative Lie algebra with the bracket $\{\cdot,\cdot\}$.

Remark 3. For a generalization to symplectic manifolds and Poisson manifolds, see [33].

3.3. **States and measurements.** The concept of a state is directly related to the conditions under which experiments are conducted. Experiments generally consist of measurements of numerical values of observables for a given system under certain definite conditions (the conditions of the experiment) and it is assumed that, at least theoretically, these conditions can be reproduced over and over again with every single measurement. However, we do not assume that different measurements give one and the same value of a given observable every time they are repeated. Rather we suppose that to give a state of the system means to prescribe the conditions of an experiment in such a way that conducting many repeated trials gives probability distributions for all the observables.

In more mathematical terms, **a state** μ on the algebra of classical observables $\mathscr{F}$ is the assignment

$$\mathscr{F} \ni f \mapsto \mu_f \in \mathcal{P}(\mathbb{R}),$$

where $\mathcal{P}(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$ and a measurement of an observable f in a state μ is the assignment

$$\mathscr{F} \times \mathscr{S} \ni (f,\mu) \mapsto \mu_f \in \mathcal{P}(\mathbb{R}),$$

where $\mathscr{S}$ is the set of all states. Hence for every Borel set $E \subset \mathbb{R}$ the quantity $0 \leq \mu_f(E) \leq 1$ is the probability that the values of the observable f in the state μ belong to E.

Taking E to be the interval $(-\infty,\lambda]$ and setting $\mu_f = \mu_f((-\infty,\lambda]) \equiv \mu_f(\lambda)$ we obtain **the distribution function** of the observable f in the state μ. We remark that the quantity $\mu_f(\lambda)$ is the probability of getting a value not exceeding λ when measuring f in the state μ. **The expectation (or the mean value)** of an observable f in a state μ is then defined by the formula

$$\langle f|\mu\rangle = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}\mu_f(\lambda).$$

Under some requirements on the mean values of observables (see [19]), it can be shown that the mean value is a positive linear functional on the algebra of classical observables $\mathscr{F}$, which, according to a general version of the Riesz representation theorem (see [28]), takes the form

$$\langle f|\mu\rangle = \int_M f(\mathbf{p},\mathbf{q})\rho_\mu(\mathbf{p},\mathbf{q}) \, \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{q},$$

where the integral is taken over the phase space M. This is the usual equivalent description of a state of a system by means of the distribution function $\rho_\mu(\mathbf{p},\mathbf{q})$ used in statistical physics.

Here ρ_μ is in general a positive generalized function. If the distribution function ρ_μ takes the form

$$\rho_\mu(\mathbf{p},\mathbf{q}) = \delta(\mathbf{p}-\mathbf{p_0})\delta(\mathbf{q}-\mathbf{q_0}),$$

then we call the state μ **a pure state.** Thus the corresponding measure on the phase space is concentrated at the point $(\mathbf{p_0},\mathbf{q_0})$ and a pure state is defined by prescribing this point of the phase space and the mean value of an observable f in a pure state μ is

$$\langle f|\mu\rangle = f(\mathbf{p_0},\mathbf{q_0}).$$

All other states that are not pure are called **mixed states.**

An important feature about mixed states is that they increase the variance of the corresponding probability distributions. Recall that **the variance** of a probability distribution is the quantity

$$\sigma_\mu^2(f) = \langle (f-\langle f|\mu\rangle)^2|\mu\rangle = \langle f^2|\mu\rangle - \langle f|\mu\rangle^2.$$

It follows that for the simplest example of a mixed state μ, which is a convex combination of two pure states μ_1 and μ_2 concentrated at the points $(\mathbf{p_1},\mathbf{q_1})$ and $(\mathbf{p_2},\mathbf{q_2})$, given by the distribution

$$\rho_\mu(\mathbf{p},\mathbf{q}) = \alpha\delta(\mathbf{p}-\mathbf{p_1})\delta(\mathbf{q}-\mathbf{q_1}) + (1-\alpha)\delta(\mathbf{p}-\mathbf{p_2})\delta(\mathbf{q}-\mathbf{q_2}),\ \ \alpha\in(0,1),$$

we have

$$\sigma_\mu^2 \geq \alpha\sigma_{\mu_1}^2 f + (1-\alpha)\sigma_{\mu_2}^2 f$$

and

$$\sigma_\mu f\sigma_\mu g \geq \alpha\sigma_{\mu_1} f\sigma_{\mu_1} g + (1-\alpha)\sigma_{\mu_2} f\sigma_{\mu_2} g$$

with equality whenever the mean values of the observables in the states μ_1 and μ_2 coincide. For a pure state μ,

$$\sigma_\mu^2 f = f^2(\mathbf{p_0},\mathbf{q_0}) - f^2(\mathbf{p_0},\mathbf{q_0}) = 0.$$

This means that for a classical system in a pure state, the result of a measurement of any observable is uniquely determined and that a state of a classical system is pure if at the time of a measurement the conditions of the experiment fix the values of all generalized coordinates and momenta.

3.4. **Time evolution of classical systems.** We close the treatment of classical mechanics with what is traditionally described as the **Hamiltonian picture** of classical mechanics. In this picture, the time evolution of a classical mechanical system is described by the system of equations of the form

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = \{H, f_t\}, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0,$$

which means that the states do not depend on time and the time dependence of the mean values of observables f in a state μ is given by the formula

$$\langle f_t|\mu\rangle = \int_M f_t(\mathbf{p},\mathbf{q})\rho(\mathbf{p},\mathbf{q})\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q} = \int_M f(G_t(\mathbf{p},\mathbf{q}))\rho(\mathbf{p},\mathbf{q})\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$$

or, in a more explicit form,

$$\langle f_t|\mu\rangle = \int_M f(\mathbf{q}(\mathbf{q_0},\mathbf{p_0},t),\mathbf{p}(\mathbf{q_0},\mathbf{p_0},t))\rho(\mathbf{q_0},\mathbf{p_0})\,\mathrm{d}\mathbf{q_0}\,\mathrm{d}\mathbf{p_0},$$

where $\mathbf{q}(\mathbf{q_0},\mathbf{p_0},t)$ and $\mathbf{p}(\mathbf{q_0},\mathbf{p_0},t)$ are the solutions of the Hamiltonian equations (12) with initial conditions $\mathbf{q}(\mathbf{q_0},\mathbf{p_0},t)\big|_{t=0} = \mathbf{q_0}$, $\mathbf{p}(\mathbf{q_0},\mathbf{p_0},t)\big|_{t=0} = \mathbf{p_0}$. If the system is in a pure state, then $\rho(\mathbf{q},\mathbf{p}) = \delta(\mathbf{q}-\mathbf{q_0})\delta(\mathbf{p}-\mathbf{p_0})$ and we have

$$\langle f_t|\mu\rangle = f(\mathbf{q}(\mathbf{q_0},\mathbf{p_0},t),\mathbf{p}(\mathbf{q_0},\mathbf{p_0},t)),$$

in accordance with the description made in (13).

Remark 4. There is an alternative but equivalent description to the Hamiltonian picture, called **the Liouville description** (or the Liouville picture) which we shall not need here, see [33].

Remark 5. For a generalization of the above results to symplectic or Poisson manifolds, see for example [2] or [33].

3.5. **Quantum mechanics.** While in classical mechanics there is always a possibility, at least theoretically, to arrange the experiment in such a way that the system is in a pure state and the corresponding variance is zero, in quantum mechanics this is no longer true. Here, even pure states lead in general to nonzero variance, which is reflected in the celebrated Heisenberg uncertainty principle and there is no chance (not even theoretically) to set up the experiment in such a way that the results of every measurement would be determined uniquely by the conditions of the experiment. This means that the realm of quantum mechanics is fundamentally different from the classical mechanics and to obtain the correct[4] mathematical framework, in which it would be possible to formulate the laws of quantum mechanic, it turns out necessarry to reject the realization of the observables as commutative algebra of functions on the phase space. Instead, one is forced to assume that to every quantum system corresponds an infinite-dimensional complex Hilbert space $\mathscr{H}$, called **the space of (pure) states** and that the system of observables $\mathscr{A}$ of a quantum system corresponding to the Hilbert space $\mathscr{H}$ is the set of all **self-adjoint operators** in $\mathscr{H}$. Finally, one has to assume that the set $\mathscr{S}$ of **states** of a quantum system with a Hilbert space $\mathscr{H}$ consists of all positive trace class operators M with $\operatorname{Tr} M = 1$. (We recall that a linear

[4]This framework, although widely accepted both by most mathematicians and physicists these days, is nevertheless doubted by some, at least in the context of its possible generalizations to quantum field theory, see for example the discussion in [27] or [13].

operator A in a separable complex Hilbert space $(\mathscr{H}, (\cdot,\cdot))$ with a domain $D(A) \subset \mathscr{H}$ is **positive** if $(A\varphi, \varphi) \geq 0$ for every $\varphi \in D(A)$; a positive linear operator A is of **trace class** (or, synonymously, belongs to the **first Schatten class**) if and only if for every complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ of the space $\mathscr{H}$

$$\sum_{j=1}^{\infty} (Ae_j, e_j) < \infty;$$

and last but not least, **the trace** of a trace class operator A is defined, independently of the choice of the orthonormal basis, by the formula

$$\operatorname{Tr} A = \sum_{j=1}^{\infty} (Ae_j, e_j)$$

and we shall denote the set of all trace class operators, which is in fact a two-sided ideal in the $\mathbb{C}^*-$algebra $\mathscr{L}(\mathscr{H})$ of bounded linear operators in $\mathscr{H}$, by the symbol $\mathscr{S}_1$.) **The pure states** of a quantum system are projection operators onto one-dimensional subspaces of $\mathscr{H}$. In particular, for the unit vectors in $\mathscr{H}$, $\psi \in \mathscr{H}$, $\|\psi\| = 1$, the corresponding projection operator onto the subspace $\mathbb{C}\psi$ is denoted by P_ψ. All other states different from the pure states are called **mixed states.**

We also define **a measurement** in a quantum system to be the assignment

$$\mathscr{A} \times \mathscr{S} \ni (A, M) \mapsto \mu_A \in \mathcal{P}(\mathbb{R}), \tag{15}$$

where, similarly to the classical case, $\mathcal{P}(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$, where for every Borel set $E \subset \mathbb{R}$, the quantity $0 \leq \mu_A(E) \leq 1$ is the probability that for a given quantum system in the state M, the result of a measurement of the observable A belongs to E. We also define the expectation value (or the mean value) of the observable $A \in \mathscr{A}$ in the state $M \in \mathscr{S}$ as

$$\langle A|M\rangle = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}\mu_A(\lambda),$$

where of course $\mu_A(\lambda) = \mu_A((-\infty, \lambda])$ is a distribution function for the probability measure μ_A.

Explicit realization of the process of measurement (15) in a quantum system is done by means of the general spectral theorem of von Neumann, which we shortly announce. To that end, we first recall the notion of **a projection-valued Borel measure** on $\mathbb{R}$. This is a mapping $\Pi : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ from the $\sigma-$algebra $\mathscr{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ into the algebra of bounded linear operators on $\mathscr{H}$ such that

(1) Π is an orthogonal projection for every $E \in \mathscr{B}(\mathbb{R})$;
(2) $\Pi(\emptyset) = 0$, $\Pi(\mathbb{R}) = I$ (the identity operator);

(3) for every countable disjoint union E of sets $E_n \in \mathscr{B}(\mathbb{R})$,

$$E = \coprod_{j=1}^{\infty} E_n,$$

$$\Pi(E) = \lim_{n\to\infty} \sum_{j=1}^{n} \Pi(E_j),$$

where the limit is taken in the strong topology on $\mathscr{L}(\mathscr{H})$.

To every projection-valued measure Π there is associated the projection-valued function

$$\Pi(\lambda) = \Pi((-\infty, \lambda]),$$

called **the projection-valued resolution of the identity** which is characterized by the following properties:

(1) $\Pi(\lambda)\Pi(\mu) = \Pi(\min\{\lambda, \mu\})$;

(2) in the sense of strong topology on $\mathscr{L}(\mathscr{H})$,

$$\lim_{\lambda\to-\infty} \Pi(\lambda) = 0, \quad \lim_{\lambda\to\infty} \Pi(\lambda) = I;$$

(3)

$$\lim_{\substack{\mu\to\lambda \\ \mu<\lambda}} \Pi(\mu) = \Pi(\lambda),$$

where the limit is again taken with respect to the strong topology on $\mathscr{L}(\mathscr{H})$.

It can be shown that for every $\psi \in \mathscr{H}$ the resolution of the identity defines a distribution function $(\Pi(\lambda)\psi, \psi)$ of bounded measure on $\mathbb{R}$, which is in fact a probability measure if $\|\psi\| = 1$.

The so called spectral theorem dating back essentially to von Neumann can be summarized as follows (see [33]):

Theorem 2 (von Neumann's spectral theorem)**.** *For every self-adjoint operator A on the Hilbert space $\mathscr{H}$ corresponding to a given quantum system there exists a unique projection-valued resolution of the identity $\Pi(\lambda) = \Pi_A(\lambda)$ such that:*

(1) *For every $\varphi \in D(A)$,*

$$A\varphi = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}\Pi(\lambda)\varphi, \tag{16}$$

where

$$D(A) = \left\{ \varphi \in \mathscr{H} : \int_{-\infty}^{\infty} \lambda^2 \, \mathrm{d}(\Pi(\lambda)\varphi, \varphi) < \infty \right\}$$

and the integral in (16) is understood as the usual spectral integral with respect to the operator-valued measure, see [7] *or* [32]*. Moreover, the support of the corresponding projection-valued measure Π_A coincides with the spectrum $\sigma(A)$ of the operator A, i.e. $\lambda \in \sigma(A)$ if and only if $\Pi_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$ for every $\varepsilon > 0$.*

(2) *For every continuous function f on $\mathbb{R}$, $f(A)$ is a linear operator on $\mathscr{H}$ defined for every $\varphi \in D(f(A))$ by*

$$f(A)\varphi = \int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}\Pi(\lambda)\varphi, \tag{17}$$

with the integral understood as in (16) and with $D(f(A))$ a dense domain in $\mathscr{H}$,

$$D(f(A)) = \left\{\varphi \in \mathscr{H} : \int_{-\infty}^{\infty} |f(\lambda)|^2\,\mathrm{d}(\Pi(\lambda)\varphi,\varphi) < \infty\right\}.$$

Moreover, for the operator $f(A)$

$$f(A)^* = \overline{f}(A),$$

where $\overline{f}$ denotes the complex conjugate function of f and the operator $f(A)$ is bounded if and only if the function f is bounded on $\sigma(A)$; for functions f and g that are bounded on $\sigma(A)$ we have for every $\varphi \in \mathscr{H}$

$$f(A)g(A)\varphi = \int_{-\infty}^{\infty} f(\lambda)g(\lambda)\,\mathrm{d}\Pi(\lambda)\varphi.$$

(3) *For every measurable function f which is in addition finite a.e. with respect to the projection-value measure Π, $f(A)$ is a linear operator on $\mathscr{H}$ defined as in (17), this time with the integral for $f(A)\varphi$ understood in the weak sense, i.e. for every $\varphi \in D(f(A))$ and for every $\psi \in \mathscr{H}$,*

$$(f(A)\varphi,\psi) = \int_{-\infty}^{\infty} f(\lambda)\,\mathrm{d}(\Pi(\lambda)\varphi,\psi), \tag{18}$$

where the integral in (18) is the usual Lebesgue-Stieltjes integral with respect to a complex measure $(\Pi(\lambda)\varphi,\psi)$ on $\mathbb{R}$ given by the polarization identity

$$\begin{aligned}(\Pi(\lambda)\varphi,\psi) &= \frac{1}{4}\{(\Pi(\lambda)(\varphi+\psi),\varphi+\psi) - (\Pi(\lambda)(\varphi-\psi),\varphi-\psi) + \\ &+ \mathrm{i}(\Pi(\lambda)(\varphi+\mathrm{i}\psi),\varphi+\mathrm{i}\psi) - \mathrm{i}(\Pi(\lambda)(\varphi-\mathrm{i}\psi),\varphi-\mathrm{i}\psi)\}\end{aligned}$$

(we recall that a measurable function f on $\mathbb{R}$ is finite a.e. with respect to the projection-valued measure Π if it is finite with respect to the measure $(\Pi\psi,\psi)$ for every $\psi \in \mathscr{H}$ and that it can be shown that, for a separable Hilbert space $\mathscr{H}$, for every projection-valued measure Π there exists $\varphi \in \mathscr{H}$ such that a function f is finite a.e. with respect to Π if and only if it is finite a.e. with respect to the measure $(\Pi\varphi,\varphi)$.) The correspondence $f \mapsto f(A)$ moreover satsifies the same properties as in the case of continuous functions, only with all the integrals understood in the weak sense.

(4) *A bounded operator B commutes with the operator A, i.e. $B(D(A)) \subset D(A)$ and $AB = BA$ on $D(A)$, if and only if B commutes with $\Pi(\lambda)$ for every λ, i.e. if and only if B commutes with every operator $f(A)$.*

(5) *For every projection-valued resolution of the identity $\Pi(\lambda)$ the operator A on $\mathscr{H}$ defined by (16) is self-adjoint.*

For a nice exposition of the spectral decomposition of operators as well as other ingredients that are present in Theorem 2, including some historical remarks, we refer to the second volume of the book [7] which has detailed proofs of most of the results listed here or to the book [32].

By means of Theorem 2 it is now possible to give an explicit description of the correspondence (15): we take $A \in \mathscr{A}$ and $M \in \mathscr{S}$ and we actually **postulate** that the assignment $(A, M) \mapsto \mu_A$ is defined by the so called **Born-von Neumann formula:**

$$\mu_A(E) = \operatorname{Tr} \Pi_A(E) M, \quad \text{for every } E \in \mathscr{B}(\mathbb{R}),$$

where Π_A is the corresponding projection-valued measure on $\mathbb{R}$ associated with the self-adjoint operator A.

The correspondence between projection-valued measures Π and self-adjoint operators given in Theorem 2 has several important consequences. For example, since positive linear operators on a complex Hilbert space are self-adjoint and since $\mathscr{S}$ is defined to be the space of positive operators of trace class, whose trace is equal to one, in particular every $M \in \mathscr{S}$ is a compact operator, we have, according to the classical Hilbert-Schmidt theorem on the decomposition of compact self-adjoint operators, that there is an orthonormal system $\{\psi_n\}$ such that

$$M = \sum_{n=1}^{N} \alpha_n P_{\psi_n}, \quad \operatorname{Tr} M = \sum_{n=1}^{N} \alpha_n = 1,$$

where $\alpha_n > 0$ are eigenvalues of M and N is understood to be equal to ∞ when the system $\{\psi_n\}$ is infinite. We thus have

$$\mu_A(E) = \sum_{n=1}^{N} \alpha_n (\Pi_A(E)\psi_n, \psi_n) = \sum_{n=1}^{N} \alpha_n \|\Pi_A(E)\psi_n\|^2 \leq \sum_{n=1}^{N} \alpha_n = 1,$$

which means that indeed $0 \leq \mu_A(E) \leq 1$ and, denoting by $\mu_A(\lambda)$ the corresponding distribution function associated to the probability measure μ_A, we have for $\psi \in \mathscr{H}$, $\|\psi\| = 1$ and for $M = P_\psi$, that $\mu_A(E) = (\Pi_A(\lambda)\psi, \psi)$.

Another important consequence of the approach taken above is that trying to generalize the Born-von Neumann formula to the case of finitely many observables $\mathbf{A} = \{A_1, \ldots, A_n\}$, we would probably expect that simultaneous measurement of a finite set of observables $\mathbf{A}$ in a state $M \in \mathscr{S}$ should result in a certain probability measure $\mu_{\mathbf{A}}$ given by the formula

$$\mu_{\mathbf{A}}(\mathbf{E}) = \operatorname{Tr}(\Pi_{A_1}(E_1) \ldots \Pi_{A_n}(E_n) M), \quad \mathbf{E} = E_1 \times \ldots \times E_n \in \mathscr{B}(\mathbb{R}^n). \tag{19}$$

It can be shown, however, that this is not the case in general. The necessary and sufficient condition for the right-hand side of (19) to define a probability measure on $\mathbb{R}^n$ is that the composition $\Pi_{A_1}(E_1) \ldots \Pi_{A_n}(E_n)$ defines a projection-valued measure on $\mathbb{R}^n$ which is in turn the case only if the projection operators Π_{A_i} commute. This is actually the case if and only if, by definition, the (probably unbounded) self-adjoint operators $A_1, \ldots, A_n$ form a commutative family. In such a case it is indeed possible to show that for such a finite set of pair-wise commuting self-adjoint opperators $\mathbf{A} = (A_1, \ldots, A_n)$ on $\mathscr{H}$ there exists the unique projection-valued measure $\Pi_{\mathbf{A}}$ on the set $\mathscr{B}(\mathbb{R}^n)$ of Borel subsets in $\mathbb{R}^n$ such that for every $\mathbf{E} = E_1 \times \ldots \times E_n \in \mathscr{B}(\mathbb{R}^n)$

$$\Pi_{\mathbf{A}}(\mathbf{E}) = \Pi_{A_1}(E_1) \ldots \Pi_{A_n}(E_n)$$

and whose support is the so called **joint spectrum** of the commutative family $\mathbf{A}$. We conclude that[5] a finite set of observables $\mathbf{A} = \{A_1, \ldots, A_n\}$ can be measured simultaneously if and only if they form a commutative family and that the simultaneous measurement of the family $\mathbf{A}$ in the state $M \in \mathscr{S}$ is described by the probability measure $\mu_{\mathbf{A}}$ whose action on Borel subset of $\mathbb{R}^n$ is given by (19).

Example 8. One of the most simplest examples of the phenomenon described in our previous discussion and at the same time one of the fundamental assumptions connecting quantum mechanics with the classical mechanics is the quantum-mechanical analogue of the motion of a single particle moving in $\mathbb{R}^n$ which somehow reflects a kind of correspondence between quantum and classical mechanics, of which the former is a deformation of a sort (we shall give a somewhat detailed treatment of this issue in Section 3.6). We take as the Hilbert space $\mathscr{H}$ the space $L^2(\mathbb{R}^n)$ (whose elements are viewed as functions of the position variables q_j) and we define the operators Q_i and P_i (the quantum analoques of coordinate functions q_i and p_i on the phase space $M = \mathbb{R}^{2n}$, called **the position and the momentum operators,** respectively) to be

$$\begin{aligned} Q_i : \psi &\mapsto q_i \psi, &(20)\\ P_i : \psi &\mapsto \frac{h}{2\pi \mathrm{i}} \frac{\partial f}{\partial q_i}. &(21) \end{aligned}$$

Of course neither of the operators Q_i and P_i maps the whole of $L^2(\mathbb{R}^n)$ into itself, since it can well happen that for $\psi \in L^2(\mathbb{R}^n)$, $q_i\psi$ is not an element of $L^2(\mathbb{R}^n)$, or ψ may fail to be differentiable at all. In fact, it is understood that the operators Q_i, P_i are defined on an appropriate dense subset of $L^2(\mathbb{R}^n)$ so that they are **unbounded operators** (by definition). Furthermore, it can be verified directly that the operators Q_i, P_i satisfy the so called **canonical commutation relations**[6]

$$\begin{aligned} [Q_i, Q_j] &= [P_i, P_j] = 0 \quad \text{for every } i, j; \\ [Q_i, P_j] &= 0 \quad \text{for } i \neq j; \\ [Q_i, P_i] &= \frac{\mathrm{i}h}{2\pi} I, \end{aligned}$$

[5] This last assertion is in fact another postulate which together with all the assumptions made above form the backbone of what is known as Dirac-von Neumann axioms of quantum mechanics, see [33].

[6] Sometimes also called Heisenberg commutation relations.

where $[A, B] = AB - BA$ is the commutator of the operators A and B. ■

In fact, the canonical commutation relations, and especially the simultaneous nonmeasurability of position and momentum operators is just a reflection of a much more general phenomenon. Indeed, if we define **the variance of the observable** A in the state M by

$$\sigma_M^2(A) = \langle (A - \langle A|M\rangle I)^2 |M\rangle = \langle A^2|M\rangle - \langle A|M\rangle^2 \geq 0,$$

provided the expectation values $\langle A^2|M\rangle$ and $\langle A|M\rangle$ exist, then we can prove the following result, the so called **(generalized) Heisenberg's uncertainty relations**:

Theorem 3. *Let $A, B \in \mathscr{A}$ and let $M = P_\psi$ be a pure state such that $\psi \in D(A) \cap D(B)$ and $A\psi, B\psi \in D(A) \cap D(B)$, then*

$$\sigma_M^2(A)\sigma_M^2(B) \geq \frac{1}{4}\langle \mathrm{i}\,[A, B]\,|M\rangle^2. \tag{22}$$

The relation (22) is just a qualitative expression of the fact that non-commuting observables cannot be measured simultaneously even if they are in a pure state.

We close this section with a quantum-mechanical analogue of the Hamiltonian picture in classical mechanics. This is the so called **Heisenberg picture** of quantum mechanics in which the states do not depend on time:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \quad \text{for } M \in \mathscr{S},$$

and bounded observables satisfy the **Heisenberg equation of motion**

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \{H, A\}_\hbar, \quad A \text{ a bounded element of } \mathscr{A} \tag{23}$$

where $\hbar$ is the reduced Planck constant, $\hbar = h/2\pi$, and $\{\cdot,\cdot\}_\hbar$ is the **quantum bracket** defined by

$$\{\cdot,\cdot\}_\hbar = \frac{\mathrm{i}}{\hbar}\,[\cdot,\cdot]\,. \tag{24}$$

Here the solutions of the equation (23) are understood by means of strongly continuous one-parameter group of unitary operators defined via the functional calculus of self-adjoint operators using Theorem 2 by

$$\mathrm{e}^{\mathrm{i}tA} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\Pi(\lambda), \quad t \in \mathbb{R},$$

where A is a self-adjoint operator on $\mathscr{H}$. In case of (23), we take the self-adjoint operator H and define

$$U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}tH}, \quad t \in \mathbb{R},$$

so that $U(t)$ satisfies the differential equation

$$\mathrm{i}\hbar\frac{\mathrm{d}U(t)}{\mathrm{d}t} = HU(t) = U(t)H$$

in the strong sense on $D(H)$ and the quantum dynamics is given by the formula

$$A(t) = U(t)^{-1} A U(t),$$

where $A(t)$ is the solution to (23) with the initial condition $A(0) = A \in \mathscr{A}_0$ (here $\mathscr{A}_0$ denotes the space of bounded observables from $\mathscr{A}$) and in this sense all quantum observables satisfy the Heisenberg equation of motion (23).

Remark 6. For an alternative (and in fact equivalent) description of the dynamics of quantum system, usually called **the Schrödinger picture**, which is a quantum analogue of the Liouville picture in classical mechanics, see [33].

3.6. **The concept of quantization.** Generally and informally speaking, quantization is a way to describe the correspondence between classical and quantum systems. This in particular means that on the usual "macroscopic" level the quantum mechanics should collapse into the classical one, which is a fundamental dictum of quantum mechanics going back to Bohr, Ehrenfest and other founders of quantum mechanics. On the mathematical side, this correspondence is expressed as a mapping $f \to Q_f$ (see the discussion done in section 3.5), which assigns to a classical observable f an (probably unbounded, self-adjoint) operator Q_f on a Hilbert space $\mathscr{H}$ in such a way that

(A) The quantum counterparts of the position and momentum coordinates q_i and p_i should be the operators Q_i and P_i given by (20) and (21).

(B) For any pair of classical observables f, g

$$Q_{f+g} = Q_f + Q_g.$$

(C) For any Borel function $\phi : \mathbb{R} \to \mathbb{R}$, if $E \subset \mathbb{R}$ and Q is a probabilistically determined quantity, the probability that $\phi(Q) \in E$ is the same as the probability that $Q \in \phi^{-1}(E)$. This means that if $Q_f = \int \lambda \, \mathrm{d}\Pi(\lambda)$, the spectral projections for $Q_{\phi \circ f}$ should be $\Pi_{\phi \circ f}(E) = \Pi_f(\phi^{-1}(E))$. These can be shown to be the spectral projections for the operator $\phi(A_f)$ defined again by means of the spectral functional calculus. Hence it should be that

$$Q_{\phi \circ f} = \phi(Q_f).$$

(D) Finally, the Bohr correspondence principle is expressed by means of the formula

$$[Q_f, Q_g] = -\frac{\mathrm{i}h}{2\pi} Q_{\{f,g\}},$$

where

$$\{f, g\} := \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right)$$

is the (canonical) Poisson bracket.

We call the domain of the mapping $Q : f \mapsto Q_f$ **the space of quantizable observables** and we would like it ideally to include at least the class of smooth functions $C^\infty(\mathbb{R}^{2n})$.

It turns out, however, that this set of requirements on the quantization procedure is severely logically inconsistent to a great extent even in the simplest case of $\mathbb{R}^{2n}$, for details on that issue see [1], [17] or [20]. One possible way out of this embarrassing situation is the so called **deformation quantization,** whose basic idea is best explained by the following important example, taken from [17]:

Example 9 (Weyl quantization)**.** Consider the phase space $\mathbb{R}^{2n}$. For sufficiently nice functions f, say $f \in \mathcal{S}(\mathbb{R}^{2n})$, the Schwartz space of smooth functions, we can write

$$f(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi, \eta)\, \mathrm{e}^{2\pi \mathrm{i}(\xi \cdot \mathbf{p} + \eta \cdot \mathbf{q})}\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{25}$$

where $\hat{f}$ is the usual Fourier transform defined for every $f \in \mathcal{S}(\mathbb{R}^n)$ by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i}\mathbf{x}\cdot\xi} f(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

with the inverse

$$\check{f}(\mathbf{x}) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\mathbf{x}\cdot\xi} f(\xi)\, \mathrm{d}\xi.$$

Regarding our previous discussion, we would like to assign to functions f a certain operator Q_f thus extending the domain of quantizable observables on $\mathbb{R}^{2n}$ from just coordinate functions q_i, p_i to "arbitrary" functions[7] on $\mathbb{R}^{2n}$. This is done by extending (25) to its "operator-valued" analogue with $\mathbf{p}$ and $\mathbf{q}$ replaced by $Q_\mathbf{p} \equiv (P_1, \ldots, P_n)$ and $Q_\mathbf{q} = (Q_1, \ldots, Q_n)$ (see (20) and (21) for the definitions of Q_i and P_i.) Indeed, from the definition of $Q_{p_i} \equiv P_i$ in (21) and from the Taylor expansion of the exponential it follows that for suitable $\psi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} &\mathrm{e}^{2\pi \mathrm{i}\xi\cdot Q_\mathbf{p}}\psi(\mathbf{q}) \\ &= \mathrm{e}^{2\pi \mathrm{i}\xi_1 Q_{p_1} + \ldots + 2\pi \mathrm{i}\xi_n Q_{p_n}}\psi(\mathbf{q}) = \psi(q_1 + h\xi_1, \ldots, q_n + h\xi_n) \\ &= \psi(\mathbf{q} + h\xi) \end{aligned}$$

and similarly

$$\mathrm{e}^{2\pi \mathrm{i}\eta\cdot Q_\mathbf{q}}\psi(\mathbf{q}) = \mathrm{e}^{2\pi \mathrm{i}\eta\cdot\mathbf{q}}\psi(\mathbf{q}).$$

In fact, expending some more extra work, one can even show that the correct substitute for $\mathrm{e}^{2\pi \mathrm{i}(\xi\cdot\mathbf{p}+\eta\cdot\mathbf{q})}$ acts like this:

$$\mathrm{e}^{2\pi \mathrm{i}(\xi\cdot Q_\mathbf{p} + \eta\cdot Q_\mathbf{q})}\psi(\mathbf{q}) = \mathrm{e}^{2\pi \mathrm{i}\eta\cdot\mathbf{q} + \pi \mathrm{i} h\eta\cdot\xi}\psi(\mathbf{q} + h\xi), \tag{26}$$

so that if we formally define

$$Q_f = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi, \eta)\, \mathrm{e}^{2\pi \mathrm{i}(\xi\cdot Q_p + \eta\cdot Q_q)}\, \mathrm{d}\xi\, \mathrm{d}\eta \equiv W_f \tag{27}$$

[7] The arbitrariness is still understood to be restricted to functions of the Schwartz class $\mathcal{S}(\mathbb{R}^{2n})$, to be precise, though extensions can be made e.g. to all tempered distributions.

(which is sometimes called the Weyl transform), we have by (26), making the change of variables $\xi \mapsto \frac{\xi - \mathbf{q}}{h}$ and using Plancherel's theorem, that

$$\begin{aligned} W_f\psi(\mathbf{q}) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi,\eta)\, \mathrm{e}^{2\pi \mathrm{i}\eta\cdot\mathbf{q}+\pi\mathrm{i}h\eta\cdot\xi}\psi(\mathbf{q}+h\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{h^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}\left(\frac{\xi-\mathbf{q}}{h},\eta\right)\mathrm{e}^{\pi\mathrm{i}\eta\cdot(\mathbf{q}+\xi)}\psi(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{h^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(\mathbf{p},\frac{\mathbf{q}+\mathbf{y}}{2}\right)\mathrm{e}^{\frac{2\pi\mathrm{i}}{h}(\mathbf{q}-\mathbf{y})\cdot\mathbf{p}}\psi(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{p}. \end{aligned}$$

It can be shown that for $f \in \mathcal{S}(\mathbb{R}^{2n})$, W_f is even continuous operator from $\mathcal{S}(\mathbb{R}^n)$ into itself and that for such f and g we have

$$W_f W_g = W_{fg} + hW_{C_1(f,g)} + O(h^2)$$

for $h \to 0$, where $C_1(f,g)$ is a multiple of the Poisson bracket given by

$$\{f,g\} := \frac{\mathrm{i}}{4\pi}\sum_{j=1}^{n}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\right),$$

for which

$$C_1(f,g) - C_1(g,f) = -\frac{\mathrm{i}}{2\pi}\{f,g\},$$

so that

$$[W_f, W_g] = -\frac{\mathrm{i}h}{2\pi}W_{\{f,g\}} + O(h^2) \quad \text{for} \quad h \to 0. \tag{28}$$

One can go even further and show that there exist bidifferential operators $C_2, C_3, \ldots$ such that

$$W_f W_g = W_{fg} + hW_{C_1(f,g)} + h^2 W_{C_2(f,g)} + h^3 W_{C_3(f,g)} + O(h^4) \quad \text{for} \quad h \to 0$$

and similarly for higher powers. The procedure of expanding $W_f W_g$ as above is sometimes written symbolically as

$$W_f W_g = W_{f*g}, \tag{29}$$

where

$$f * g = fg + hC_1(f,g) + h^2 C_2(f,g) + h^3 C_3(f,g) + \ldots, \tag{30}$$

a formal power series in h with the understanding that (30) is just an asymptotic expansion

$$W_f W_g = \sum_{j=0}^{N-1} h^j W_{C_j(f,g)} + O(h^N) \quad \text{as} \quad h \to 0$$

for every $N = 0, 1, 2 \ldots$. ■

The point here is that if we somewhat weaken our quantization requirements (A) to (D) in such a way that we retain the canonical quantization (A), the linearity (B), and the correspondence principle (D) only in an appropriate approximate sense:

$$[Q_f, Q_g] = -\frac{\mathrm{i}h}{2\pi}Q_{\{f,g\}} + O(h^2) \quad \text{as} \quad h \to 0,$$

and if we discard the condition (C) with the sole exception that we do require $Q_{\mathbf{1}} = I$, where $\mathbf{1}$ is the constant function one and I is the identity operator, then the Weyl transform is in fact a special instance of the deformation quantization described as follows: consider a (symplectic or, more generally, Poisson) manifold $\mathcal{M}$ and the ring $C^\infty(\mathcal{M})[\![h]\!]$ of formal power series with coefficients in $C^\infty(\mathcal{M})$. We define **a star-product** to be an associative mapping $*$ such that

$$f * g = \sum_{j=0}^{\infty} h^j C_j(f,g) \quad \text{for every } f, g \in C^\infty(\mathcal{M}),$$

where the operators C_j satisfy the identities

$$C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = -\tfrac{\mathrm{i}}{2\pi}\{f,g\},$$
$$C_j(f,\mathbf{1}) = C_j(\mathbf{1},f) = 0 \quad \text{for every } j \geq 1$$

and the mapping $*$ is $\mathbb{C}[\![h]\!]-$bilinear. In Example 9 the corresponding star-product is given by (29).

Remark 7. For a connection of the Weyl correspondence with the theory of pseudodifferential operators due to Kohn and Nirenberg, see for example [23], [21] or [20] and references therein.

Despite the fact that the Weyl correspondence is probably the first general quantization procedure ever invented and in certain respects also one of the most satisfactory ones, its major drawback is that it is narrowly connected to $\mathbb{R}^{2n}$ by means of the Fourier transform.

A second objection goes to the fact that the deformation quantization as sketched above is only a formal procedure without any assumptions on convergence of the formal power series.

To obtain a generalization to other domains, as well as a way to construct star-products that are somehow connected with the geometry and analysis of the given manifold, we can resort to an idea going back to Berezin [8] and employ the concept of weighted Bergman spaces introduced in Section 2. This approach is illustrated in the next section using certain special submanifolds in $\mathbb{C}^n$.

4. Berezin quantization

In this section, we closely follow [17]. Consider the weighted Bergman space $A^2(\Omega,\rho)$ with Ω a domain in $\mathbb{C}^n$ such that the (positive and continuous) weight function ρ is integrable with respect to the Lebesgue measure λ (this last minor technical assumption is required for the weighted Bergman kernel $K_\rho(x,y)$ to satisfy $K_\rho(x,x) = \|K_{\rho,x}\|^2 > 0$.) We define the Berezin symbol $\widetilde{T}$ of an operator T on $A^2(\Omega,\rho)$ to be a function on Ω defined by

$$\widetilde{T}(x) = \frac{\langle TK_{\rho,x}, K_{\rho,x}\rangle}{K_{\rho,x}, K_{\rho,x}} = \langle Tk_{\rho,x}, k_{\rho,x}\rangle \text{ for every } x \in \Omega, \text{ where } k_{\rho,x} = \frac{K_{\rho,x}}{\|K_{\rho,x}\|}.$$

We remark that it can be quite easily verified that the mapping $T \mapsto \widetilde{T}$ is one-to one.

No less important is **the Toeplitz operator** on $A^2(\Omega, \rho)$ defined for every $\phi \in L^\infty(\Omega)$ by

$$T_\phi f = P_\rho(\phi f),$$

where P_ρ is the weighted Bergman projection $P_\rho : L^2(\Omega, \rho) \to A^2(\Omega, \rho)$ (in fact an orthogonal projection onto $A^2(\Omega, \rho)$). Summing up, we have a mapping assigning an operator T_f acting on $A^2(\Omega, \rho)$ to a bounded function f and a mapping assigning a function $\widetilde{T_f}$ to the operator T_f. The composition of these two mappings is denoted by $B_\rho f$ and called **the Berezin transform.** We have (for every $f \in L^\infty(\Omega)$)

$$B_\rho f = \widetilde{T_f},$$

or equivalently

$$B_\rho f(x) = \frac{\langle f K_{\rho,x}, K_{\rho,x}\rangle}{K_{\rho,x}, K_{\rho,x}} = \int_\Omega f(y) \, \frac{|K_\rho(x,y)|^2}{K_\rho(x,x)} \, \rho(y) \, \mathrm{d}y.$$

Now for the weights ρ and domains Ω as discussed above, and due to the injectivity of the map $T \mapsto \widetilde{T}$, we can identify operators with functions and define a (clearly noncommutative) product $*_\rho$ as a binary operation on a certain set $\mathcal{A}_\rho$ of functions on Ω defined by

$$\widetilde{S} *_\rho \widetilde{T} = \widetilde{ST},$$

where $\widetilde{S}$ is the Berezin symbol of an operator S. Hence (see the definition of the Berezin symbol) the product $f *_\rho g$ is defined only for those f and g that can be written as Berezin symbols of certain operators on $A^2(\Omega, \rho)$, which means that in fact

$$\mathcal{A}_\rho = \left\{ \widetilde{T} : T \text{ is a continuous linear operator on } A^2(\Omega, \rho) \right\}.$$

The idea of **the Berezin quantization** is to find a family of weights $\rho = \rho(h) \equiv \rho_h$ such that

$$\mathcal{A} = \bigcap_{h>0} \mathcal{A}_{\rho_h}$$

is big enough for certain bidifferential operators $C_j(\cdot, \cdot)$ to be uniquely determined by their values on functions $f, g \in \mathcal{A}$ and the C_j's are such that for every $f, g \in \mathcal{A}$

$$f *_{\rho_h} g = \sum_{j=0}^{\infty} h^j C_j(f, g)$$

asymptotically for $h \to 0$, where $C_0(f, g) = fg$ and

$$C_1(f, g) - C_1(g, f) = \frac{\mathrm{i}}{2\pi} \{f, g\}$$

for a given Poisson bracket $\{\cdot, \cdot\}$ on Ω.

Then it can be shown that the bidifferential operators C_j define a star-product

$$f * g = \sum_{j=0}^{\infty} h^j C_j(f, g) \tag{31}$$

for every $f, g \in C^\infty(\Omega)$, called **the Berezin star-product**. In fact, the problem of finding appropriate weights ρ_h reduces to a problem of finding the weights ρ_h in such a way that the associated Berezin transforms $B_{\rho_h} \equiv B_h$ have an asymptotic expansion of the form

$$B_h = Q_0 + hQ_1 + h^2 Q_2 + \dots, \quad \text{for} \quad h \to 0, \tag{32}$$

with Q_j being certain differential operators such that $Q_0 = I$, the identity operator. In such a case, writing

$$Q_j(f) = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f$$

and setting

$$f *_B g = \sum_{j=0}^\infty h^j C_j(f, g),$$

where

$$C_j(f, g) = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta} \left(\bar{\partial}^\beta f\right) (\partial^\alpha g),$$

it can be shown that if

$$C_1(f, g) - C_1(g, f) = \frac{\mathrm{i}}{2\pi}\{f, g\},$$

then $f *_B g$ coincides with the Berezin star-product (31) for every $f, g \in C^\infty(\Omega)$.

One of the main results on the existence of the appropriate weights for which the expansion (32) is valid and which at the same time takes into account the geometric properties of the given domain Ω is the following:

Theorem 4 (see [17]). *Let $\Omega \subset \mathbb{C}^n$ be smoothly bounded and strictly pseudoconvex, and Φ a strictly plurisubharmonic function on Ω such that $\mathrm{e}^{-\Phi} = r$ is a defining function for Ω. Then for the weights $w = \mathrm{e}^{-\alpha\Phi}\det(\partial\bar{\partial}\Phi)$, we have as $\alpha \to +\infty$, $\alpha \in \mathbb{Z}$,*

$$K_\alpha(x, x) \sim \mathrm{e}^{\alpha\Phi(x)} \frac{\alpha^n}{\pi^n} \sum_{j=0}^\infty \frac{b_j(x)}{\alpha^j}$$

with certain functions $b_j \in C^\infty(\Omega)$, $b_0 = \det(\partial\bar{\partial}\Phi)$ and

$$B_\alpha f = \sum_{j=0}^\infty \frac{Q_j f}{\alpha^j},$$

where Q_j are some differential operators with $Q_0 = I$ and

$$Q_1 = \sum_{j,k=1}^n g^{\bar{j}k} \frac{\partial^2}{\partial z_k \partial \bar{z}_j} = \Delta,$$

the Laplace-Beltrami operator, and $g^{\bar{j}k}$ is a matrix inverse to the matrix $g_{j\bar{k}} = \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}$.

We recall that a bounded domain $\Omega \subset \mathbb{C}^n$ with smooth boundary is **strictly pseudoconvex**[8] if there exists a smooth function $r : \Omega \to \mathbb{R}$, called **a (strictly plurisubharmonic) defining function** for Ω, such that

(i) $r > 0$ on Ω,
(ii) $r = 0,\ \|\nabla r\| > 0$ on $\partial\Omega$,
(iii) $-r$ is a strictly plurisubharmonic function in a neighbourhood of $\operatorname{cl}\Omega$;

(a smooth function $\Phi : \Omega \to \mathbb{R}$ is **strictly plurisubharmonic** if for every $z \in \Omega$ and every $v \in \mathbb{C}^n$ the function $\phi : t \mapsto \Phi(z + tv)$, $t \in \mathbb{C}$, is strictly subharmonic wherever defined. For a good treatment of the concept of subharmonic and plurisubharmonic functions, see [25].)

Remark 8. For the sake of simplicity the whole discussion in this section has been done for Ω a domain in $\mathbb{C}^n$. It turns out that all this machinery can be extended even to general Kähler manifolds with certain technical adjustments, see for example [1]. For further information on other types of domains, where the expansion (32) is actually true, see also [15] and references therein.

5. The harmonic Berezin transform - known results

As we have seen, all the main ingredients in the Berezin quantization procedure as described briefly in Section 4, have been hitherto intimately connected solely with the (weighted) *holomorphic* Bergman spaces. What is not that clear is why should holomoprhic functions have anything in common with quantization procedures and a natural question therefore is whether also other reproducing kernel Hilbert spaces could be used enabling one to ultimately extend the quantization procedures not only to Kähler manifolds but also to symplectic manifolds, say.

Here, the situation gets quite convoluted. On one hand, it turns out that instead of spaces of holomorphic functions one could indeed equally well use the spaces of square-integrable *harmonic* functions mentioned already in Section 2.2 or, even more generally, spaces of square-integrable functions L^2_A of functions that are annihilated by a given fixed hypoelliptic partial differential operator A (of which both holomorphic and harmonic Bergman spaces are special instances) for general operator symbols as well as Toeplitz operators to make sense, see [15].

On the other hand, it can be proved at the same time that the Berezin quantization procedure generally breaks down even in the case of harmonic Bergman spaces since the correspondence between operators and their Berezin symbols is not one-to-one anymore, see [15] again.

[8]In general, pseudoconvex domains are natural domains for holomorphic functions in $\mathbb{C}^n$, see [25] and the related discussion therein.

That much being said, the more interesting seems to be that asymptotic expansions of the form (32) are available also in the context of some harmonic Bergman spaces as we will try to illustrate in the present section.

As a first example, it has been proved in [16] that for the *harmonic* (rather than holomorphic) Segal-Bargmann (Fock) space on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, i.e. the space

$$\mathcal{H}_h := \{f \in L^2(\mathbb{R}^{2n}, \rho) : \Delta f = 0\},$$

introduced in Example 6, there is the following result:

Theorem 5 (see [16])**.** *Let $\mathcal{R}$ be an operator given by the formula*

$$\mathcal{R} = \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right) = \sum_{j=1}^{m} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right),$$

where $z_j = x_j + \mathrm{i}y_j$, acting on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Then there are linear differential operators $R_0, R_1, \ldots$ on $\mathbb{R}^{2n} \setminus \{0\}$ of the form

$$R_j = \sum_{\substack{k,l \geq 0 \\ k+2l \leq 2j}} \rho_{jkl} |y|^{2l-2j} \mathcal{R}^k \Delta^l$$

with some constants ρ_{jkl} (depending only on n), such that for any $y \neq 0$ and for any $f \in L^\infty(\mathbb{R}^{2n})$ smooth in a neighbourhood of y, the harmonic Berezin transform B_h^{harm} associated to the spaces $\mathcal{H}_h$ has the asymptotic expansion

$$B_h^{\mathrm{harm}} f(y) \sim \sum_{j=0}^{\infty} \frac{R_j f(y)}{\alpha^j} \quad \text{as } \alpha \to \infty. \tag{33}$$

Furthermore, R_0 is the identity operator,

$$R_1 = \frac{\Delta}{4(2n-1)} + \frac{(4n-3)(n-1)}{2(2n-1)|y|^2}\mathcal{R} + \frac{n-1}{2(2n-1)|y|^2}\mathcal{R}^2,$$

and

$$B_h^{\mathrm{harm}} f(0) \sim \sum_{j=0}^{\infty} \frac{\Delta^j f(0)}{j!(4\alpha)^j} \quad \text{as } \alpha \to \infty. \tag{34}$$

Note that (33) doesn't actually reduce to (34) in case $y = 0$, (where even the operator R_1 is singular) which suggests that B_h^{harm} enjoys quite an abruptive behaviour at $y = 0$ (this is in fact a manifestation of the so called Stokes phenomenon).

As a second example of an expansion akin to (32), we take the harmonic Bergman space of functions on the unit ball in $\mathbb{R}^n$ from Example 7. It has been shown in [11] that for the corresponding Berezin transform

$$(B_\alpha f)(x) := \int_{\mathbb{B}^n} f(y) \frac{K_\alpha^2(x,y)}{K_\alpha(x,x)} \mathrm{d}\mu_\alpha^n(y), \tag{35}$$

where

$$\mathrm{d}\mu_\alpha^n(y) := c_\alpha (1 - |y|^2)^\alpha \mathrm{d}^n y, \quad \alpha > -1,$$

with

$$c_\alpha = \frac{\Gamma\left(\alpha + \frac{n}{2} + 1\right)}{\pi^{n/2}\alpha!},$$

the following result is true:

Theorem 6 (see [11])**.** *For $x \neq 0$, $n > 1$, and $f \in C^\infty(\mathbb{B}^n)$, there exist differential operators $Q_i := Q_i\left(\Delta, x\cdot\nabla, |x|^2\right)$, involving only the Laplace operator Δ, the directional derivative $x\cdot\nabla$ and the quantity $|x|^2$, such that*

$$\left(B_\alpha f\right)(x) = \int\limits_{\mathbb{B}^n} f(y)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}\mathrm{d}\mu_\alpha^n(y) \sim \sum_{i=0}^\infty \frac{Q_i f(x)}{\alpha^i} \qquad (\alpha\to\infty),$$

where $Q_0 = I$ and

$$Q_1 = \frac{n-2}{2}\frac{1-|x|^2}{|x|^2}x\cdot\nabla + \frac{(n-2)(1-|x|^2)^2}{4(n-1)\,|x|^2}\left(x\cdot\nabla\right)^2 + \frac{1}{4(n-1)}(1-|x|^2)^2\Delta.$$

Moreover, for $x = 0$ it holds

$$\left(B_\alpha f\right)(0) \sim \sum_{i=0}^\infty \frac{\Delta^i f(0)}{4^i\left(\alpha+\frac{n}{2}+1\right)_i} \qquad (\alpha\to\infty).$$

Again, also in Theorem 6 a kind of Stokes phenomenon is clearly present for $x = 0$. In the next section we describe yet another result of this kind for the harmonic Bergman space of functions on the half-space.

6. Asymptotic expansion of the harmonic Berezin transform on the half-space

In the paper [22], constituting a part of the thesis, we are addressing the question of whether the harmonic analogue of the expansion (32), corresponding to the weighted harmonic Bergman space $B^2(H,\rho)$, where H is the upper half-space $\{(x,y)\in\mathbb{R}^n\times\mathbb{R} : y>0\}$ and $\rho(x,y) = y^\alpha$, where $\alpha > -1$, holds true.

It turns out that, surprisingly or not, such an expansion is indeed available and this is the main result of [22]. More explicitly, denoting the corresponding Berezin transform by B_α :

$$B_\alpha f(x,y) = \frac{1}{K_\alpha(x,y;x,y)}\int_{\mathbb{R}^n}\int_0^\infty f(z,w)|K_\alpha(x,y;z,w)|^2\; w^\alpha\,\mathrm{d}w\,\mathrm{d}z,$$

where K_α is the reproducing kernel of H and $f \in L^\infty(H) = L^\infty(H,\rho)$ we have the following:

Theorem 7 (see [22])**.** *For any $f \in L^\infty(H)\cap C^\infty(H)$,*

$$\left(B_\alpha f\right)(x,y) \sim \sum_{j=0}^\infty \frac{R_j f(x,y)}{\alpha^j} \tag{36}$$

as $\alpha \longrightarrow +\infty$ *for every* $(x,y) \in H$*, where* R_j *are certain differential operators, with* $R_0 f(x,y) = f(x,y)$ *(*R_0 *thus being the identity operator) and*

$$R_1 f(x,y) = y^2 \frac{\Delta f(x,y)}{n} + (1-n) y \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y),$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

The proof of Theorem 7 is done in several steps. First of all, we note that the symmetry of H both with respect to horizontal translations as well as to dilations leaves the harmonic functions invariant under these special transformations which has the effect that upon denoting $f^{a,b}(x,y) := f(bx+a, by)$ (here $a \in \mathbb{R}$ and $b > 0$) we obtain that for any $f \in L^\infty(H)$,

$$(B_\alpha f)(a,b) = (B_\alpha f^{a,b})(0,1).$$

This in turn shows that to prove Theorem 7 it is in fact enough to prove that for every $f \in L^\infty(H)$

$$(B_\alpha f)(0,1) \sim \sum_{k=0}^{\infty} \frac{Q_k f(0,1)}{\alpha^k}, \quad \alpha \longrightarrow +\infty, \tag{37}$$

where $Q_k f(x,y) = Q_k f^{x,y}(0,1)$ for every $(x,y) \in H$.

At this point, using the abbreviation $H_\alpha(x,y) := K_\alpha(x,y;0,1)$, we show that upon performing Fourier transform with respect to the $x-$variable, there is the following integral representation of H_α :

$$H_\alpha(x,y) = \frac{1}{(2\pi)^n \Gamma(\alpha+1)} \int_{\mathbb{R}^n} (2|\xi|)^{\alpha+1} \mathrm{e}^{-(y+1)|\xi|} \mathrm{e}^{i\xi\cdot x} \,\mathrm{d}\xi,$$

or, in spherical coordinates, $\xi = r\zeta$, $r > 0$, $\zeta \in \mathbf{S}^{n-1}$,

$$H_\alpha(x,y) = \frac{2^{\alpha+1}}{(2\pi)^n \Gamma(\alpha+1)} \int_0^\infty \int_{\mathbf{S}^{n-1}} r^{\alpha+n} \mathrm{e}^{-(y+1)r} \mathrm{e}^{\mathrm{i}r\zeta\cdot x} \,\mathrm{d}\sigma(\zeta)\,\mathrm{d}r.$$

Transforming the last integral slightly via the changes $r \mapsto \frac{r}{y+1}$ and $r \mapsto \alpha r$ yields

$$H_\alpha(x,y) \quad = \quad \frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n \Gamma(\alpha+1)(y+1)^{\alpha+n+1}\mathrm{e}^\alpha} \int_0^\infty \int_{\mathbf{S}^{n-1}} r^{\alpha+n} \mathrm{e}^{(1-r)\alpha} \mathrm{e}^{\mathrm{i}\frac{r\alpha\zeta\cdot x}{y+1}} \,\mathrm{d}\sigma(\zeta)\,\mathrm{d}r. \tag{38}$$

In the form (38), the kernel H_α already helps us to fix the expansion (37). Indeed, if we plug (38) into the very definition of the Berezin transform, we can write

$$B_\alpha f(0,1) = \frac{1}{H_\alpha(0,1)} \left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n \Gamma(\alpha+1)\mathrm{e}^\alpha} \right)^2 \int_0^\infty \int_{\mathbf{R}^n} f(z,w) \frac{w^\alpha}{(w+1)^{2\alpha+2n+2}}$$
$$\times \int_0^\infty \int_0^\infty \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} r^n s^n \left(\frac{r}{\mathrm{e}^{r-1}} \frac{s}{\mathrm{e}^{s-1}} \right)^\alpha \mathrm{e}^{\frac{\mathrm{i}\alpha(r\zeta - s\eta)\cdot z}{w+1}} \,\mathrm{d}\sigma(\zeta)\,\mathrm{d}\sigma(\eta)\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}w. \tag{39}$$

It is a consequence of a well-known result from harmonic analysis on the existence of left and right invariant Haar measure g on the orthogonal group $O(n)$ (which is in fact a compact

Lie group) that for every function F that is continuous on $\mathbf{S}^{n-1}$,

$$\int_{\mathbf{S}^{n-1}} F(\zeta)\, \mathrm{d}\sigma(\zeta) = \omega_n \int_{O(n)} F(ge_1)\, \mathrm{d}g,$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the total volume of the sphere, $g \in O(n)$ and $e_1 = (1, 0, \ldots, 0) \in \mathbf{S}^{n-1}$. This observation enables us to finally rewrite the integral (39) in the form

$$\begin{aligned} B_\alpha f(0,1) &= \frac{\omega_n}{H_\alpha(0,1)} \left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n \Gamma(\alpha+1) \mathrm{e}^\alpha} \right)^2 \int_0^\infty \int_0^\infty \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{f^*(|z|, w)|y|s^n}{(w+1)^{2n+2}} \\ &\quad \left(\frac{w}{(w+1)^2} \frac{|w|}{\mathrm{e}^{|w|-1}} \frac{s}{\mathrm{e}^{s-1}} \mathrm{e}^{\frac{i(y-se_1)\cdot z}{w+1}} \right)^\alpha \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}s\, \mathrm{d}w, \end{aligned} \tag{40}$$

where $f^*(t, w) := \int_{O(n)} f(gte_1, w)\, \mathrm{d}g$. This is in fact an integral of the form

$$\mathcal{I}(\alpha) = \int_\Omega F(x) \mathrm{e}^{\alpha S(x)}\, \mathrm{d}x,$$

to which, as it is possible to verify, the standard multidimensional Laplace method of asymptotic expansions of integrals of this type can be applied:

$$\mathcal{I}(\alpha) \sim \left(\frac{2\pi}{\alpha} \right)^{\frac{\dim \Omega}{2}} \frac{\mathrm{e}^{\alpha S(x_0)}}{|\text{Hess } S(x_0)|^{1/2}} \sum_{j=0}^{\infty} \left(\sum_{k=j}^{3j} \frac{1}{k!(k-j)!2^k} L_S^k (S(x, x_0)^{k-j} F(x)) \big|_{x=x_0} \right) \alpha^{-j},$$

as $\alpha \longrightarrow +\infty$ (here Hess $S(x_0)$ is the determinant of the matrix

$$A = -\left(\frac{\partial^2 S(x_0)}{\partial x_j x_k} \right)_{j,k=1}^{\dim \Omega}, \tag{41}$$

L_S is the constant-coefficent differential operator on Ω given by the formula

$$L_S = \sum_{j,k=1}^{\dim \Omega} (A^{-1})_{j,k} \frac{\partial^2}{\partial x_j x_k},$$

where A^{-1} is the inverse of the matrix (41), and

$$S(x, x_0) := S(x) - S(x_0) + \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle).$$

Thus, in the context of the integral (40), taking $\Omega := \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$, $x = (w, s, z, y) \in \Omega$,

$$F(x) := \frac{f^*(|z|, w)|y|s^n}{(w+1)^{2n+2}}$$

and

$$S(x) := \ln \frac{w}{(w+1)^2} + (\ln s + 1 - s) + \frac{\mathrm{i}(w - se_1) \cdot z}{w+1} + (\ln|y| + 1 - |y|),$$

we obtain the asymptotic expansion

$$B_\alpha f(0,1) \sim c_{\alpha,n} \sum_{j=0}^{\infty} \left(\sum_{k=j}^{3j} \frac{1}{k!(k-j)!2^k} L_S^k (S(x, x_0)^{k-j} F(x)) \big|_{x=x_0} \right) \alpha^{-j},$$

where

$$c_{\alpha,n} := \frac{\omega_n}{H_\alpha(0,1)} \left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n \Gamma(\alpha+1) \mathrm{e}^\alpha} \right)^2 \left(\frac{2\pi}{\alpha} \right)^{n+1} \frac{\mathrm{e}^{\alpha S(x_0)}}{|\text{Hess } S(x_0)|^{1/2}}.$$

This is almost what we need and after we unwrap the corresponding definitions and after a substantial deal of computation, we do indeed arrive at the desired formula

$$B_\alpha f(0,1) \sim \sum_{k=0}^\infty \frac{Q_k f(0,1)}{\alpha^k}, \quad \alpha \longrightarrow +\infty,$$

with $Q_0 f(0,1) = f(0,1)$, so that $Q_0 f(x,y) = Q_0 f^{x,y}(0,1) = f^{x,y}(0,1) = f(x,y)$, and $Q_1 f(0,1) = \frac{(\Delta_x f)(0,1)}{n} + (1-n)\frac{\partial f}{\partial y}(0,1) + \frac{\partial^2 f}{\partial y^2}(0,1)$, so that $R_1 f(x,y) = R_1 f^{x,y}(0,1) = y^2 \frac{\Delta f}{n}(x,y) + (1-n) y \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y)$ for every $(x,y) \in H$.

Remark 9. We note that, unlike for the harmonic Fock space or for the unit ball in $\mathbb{R}^n$, no Stokes phenomenon is present in the assertion of Theorem 7.

7. Berezin transform of two arguments

In the paper [12], which forms the second part of the thesis, we study the asymptotic properties analogous to (32) of a "natural" extension of the Berezin transform of functions bounded on a suitable domain Ω, treating it as a restriction to the diagonal $x = z$ of a function of *two* arguments defined by

$$B_\alpha^2 f(x,z) = \frac{\langle f K_{\alpha,z}, K_{\alpha,x} \rangle}{\langle K_{\alpha,z}, K_{\alpha,x} \rangle} = \int_\Omega f(y) \frac{K_\alpha(x,y) K_\alpha(y,z)}{K_\alpha(x,z)} \rho_\alpha(y) \, \mathrm{d}y, \tag{42}$$

the right-hand side of (42) being of course defined whenever $K_\alpha(x,z) \neq 0$.

The paper [12] consists of two parts. In the first part we deal with the transform (42) in the context of holomorphic Bergman spaces. To be more specific, we consider the Segal-Bargmann or Fock space $\mathcal{F}_\alpha$ of all entire functions in $\mathbb{C}^n$ that are square-integrable with respect to the measure

$$\mu_\alpha^{2n}(y,\bar{y}) = \left(\frac{\alpha}{\pi} \right)^n e^{-\alpha|y|^2} \lambda(y),$$

where $\lambda(y)$ is the usual $2n$-dimensional Lebesgue volume measure with the factor $(\alpha/\pi)^n$ chosen so that the whole space is of measure one.

As we know from Example 4, the corresponding Bergman kernel K_α is the mapping $\mathbb{C}^n \times \mathbb{C}^n \ni (x,y) \mapsto K_\alpha(x,y) = e^{\alpha x \cdot \bar{y}} = e^{\alpha(x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n)} \in \mathbb{C}$.

The main result of the first part, then, is the following: for the holomorphic Berezin transform $B_\alpha^2 f(x,z)$ of two arguments defined for f depending on two variables by the formula

$$B_\alpha^2 f(x,z) = \int_\Omega f(y,\overline{y}) \frac{K_\alpha(x,y) K_\alpha(y,z)}{K_\alpha(x,z)} \, \mathrm{d}\mu_\alpha^{2n}(y,\overline{y}),$$

we have the following

Theorem 8 (see [12])**.** *Let f be a polynomial on $\mathbb{C}^n \times \mathbb{C}^n$. Then, as $\alpha \longrightarrow \infty$,*

$$(B_\alpha^2 f)(x,z) \sim f(x,\bar{z}) + \frac{\partial_x \bar{\partial}_z f(x,\bar{z})}{\alpha} + \frac{(\partial_x \bar{\partial}_z)^2 f(x,\bar{z})}{\alpha^2 2!} + \dots \tag{43}$$

The proof of Theorem 8 is quite straightforward and is done essentially by reducing it to the case of the usual Berezin transform, we refer to [12] for the details.

Remark 10. Note that in case $z = x$, the expansion (43) is the same as the usual expansion in (32).

The second part concerns the, admittedly much harder, case of harmonic functions: we consider the harmonic Fock space $\mathcal{H}_\alpha$ of all harmonic functions in $\mathbb{R}^n$ that are square-integrable with respect to the measure

$$\mu_\alpha^n(y) := c_\alpha e^{-\alpha|y|^2} \lambda^n(y),$$

where the factor $c_\alpha := \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}}$ is again chosen so that the whole space is of measure one and $\lambda^n(y)$ is the Lebesgue volume measure on $\mathbb{R}^n$.

As we already know from Example 6, the corresponding harmonic Bergman kernel $R_\alpha(x,y)$ in this case is given by the formula

$$R_\alpha(x,y) = \Phi_2\left(\begin{matrix} - \\ b \end{matrix} ; \begin{matrix} b & b \\ & - \end{matrix} ; \alpha u_{x,y}, \alpha \bar{u}_{x,y}\right),$$

where $b := \frac{n}{2} - 1$, $u_{x,y} = x \cdot y + \mathrm{i}\sqrt{|x|^2 |y|^2 - (x \cdot y)^2}$ and Φ_2 is a hypergeometric function of two variables from the Horn's list (see, [5],[31]):

$$\Phi_2\left(\begin{matrix} - \\ c \end{matrix} ; \begin{matrix} b_1 & b_2 \\ & - \end{matrix} ; x, y\right) = \sum_{j,k=0}^{\infty} \frac{(b_1)_j (b_2)_k}{(c)_{j+k}} \frac{x^j y^k}{j!k!}, \qquad \forall x, y \in \mathbb{C}. \tag{44}$$

We then show that the asymptotic behaviour (in its principal term) and, accordingly, the appropriate "limiting point" v, for which (in analogy with the expansion (32))

$$(B_\alpha^2 f)(x,z) \longrightarrow f(v), \qquad \alpha \longrightarrow \infty,$$

where $(B_\alpha^2 f)(x,z)$ is the harmonic Berezin transform of two arguments defined[9] by the formula

$$(B_\alpha^2 f)(x,z) := \int_{\mathbb{R}^n} f(y) \frac{R_\alpha(x,y) R_\alpha(z,y)}{R_\alpha(x,z)} \mathrm{d}\mu_\alpha^n(y) = (B_\alpha^2 f)(z,x), \tag{45}$$

are as in the following two theorems, which naturally complement each other:

[9]For certain technical reasons, we are forced to restrict ourselves, as it was actually the case already in Theorem 8, to polynomial functions f on $\mathbb{R}^n$. See also [12] for a somewhat more detailed discussion of this matter.

Theorem 9 (see [12]). *Let p be a polynomial on $\mathbb{R}^n$, $x, z \in \mathbb{R}^n$ not collinear. Then for $\alpha \in \mathbb{C}$ such that* $\operatorname{Re}(\alpha u_{x,z}) > \operatorname{Re}(\alpha \bar{u}_{x,z})$, $\operatorname{Re}(\alpha u_{x,z}) > 0$, $\operatorname{Re}(\alpha) > 0$, *we have*

$$(B_\alpha^2 p)(x,z) \longrightarrow p(v), \qquad |\alpha| \longrightarrow \infty,$$

where

$$v = v_{x,z} := x\frac{u_{x,z} - |z|^2}{u_{x,z} - \bar{u}_{x,z}} + z\frac{u_{x,z} - |x|^2}{u_{x,z} - \bar{u}_{x,z}}, \qquad u_{x,z} := x \cdot z + \mathrm{i}\sqrt{|x|^2 |z|^2 - (x \cdot z)^2},$$

and the point $v \in \mathbb{C}^n$ moreover satisfies the following relations:

$$\operatorname{Re} v = \frac{x+z}{2}, \qquad v \cdot \bar{v} = |x+z|^2 + |x-z|^2, \qquad v \cdot v = u_{x,z}, \qquad x \cdot v = \frac{|x|^2 + u_{x,z}}{2},$$

$$(x \cdot v)(\overline{x \cdot v}) = \frac{|x|^2 |x+z|^2}{4}, \qquad \bar{u}_{x,z}(x \cdot v) = |x|^2 (\overline{z \cdot v}), \qquad (x \cdot v)(z \cdot v) = \frac{u_{x,z} |x+z|^2}{4}.$$

For $\alpha \in \mathbb{R}$ and $\nabla p \neq 0$, the limit does not exist.

Theorem 10 (see [12]). *Let p be a polynomial on $\mathbb{R}^n$, $\xi, \alpha > 0$, $u_{x,t}$ as above. Then*

(1) *For $z = \xi x$,*

$$(B_\alpha^2 p)(x, \xi x) \longrightarrow p(\nabla_t) e^{x \cdot t} \Phi_2 \left(\begin{matrix} - \\ n-2 \end{matrix} ; \begin{matrix} \frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{matrix} ; \frac{\xi-1}{2} u_{x,t}, \frac{\xi-1}{2} \bar{u}_{x,t} \right) \Bigg|_{t=0}, \qquad \alpha \longrightarrow \infty.$$

(2) *For $z = 0$,*

$$(B_\alpha^2 p)(x, 0) \longrightarrow p(\nabla_t) \Phi_2 \left(\begin{matrix} - \\ \frac{n}{2}-1 \end{matrix} ; \begin{matrix} \frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{matrix} ; \frac{1}{2} u_{x,t}, \frac{1}{2} \bar{u}_{x,t} \right) \Bigg|_{t=0}, \qquad \alpha \longrightarrow \infty.$$

(3) *For $z = -\xi x$, For $p(y) = p_1(y, |y|^2)$, where p_1 is a linear function in the first argument and a polynomial in the other, we have*

$$\begin{aligned} (B_\alpha^2 p)(x, -\xi x) &\longrightarrow p_1\left(x\frac{1-\xi}{2}, -\xi |x|^2 \right), \qquad n > 3 \text{ even}, \\ (B_\alpha^2 p)(x, -\xi x) &\longrightarrow p(0), \qquad n > 1 \text{ odd}, \\ \left| (B_\alpha^2 (y_1 y_2))(x, -\xi x) \right| &\longrightarrow \infty, \qquad \text{all } n > 2. \end{aligned}$$

Remark 11. As seen from the statements of the theorems we were able to supply their proofs only for polynomials. It is not clear to the authors what techniques are to be used to prove the corresponding result also for entire integrable functions on $\mathbb{C}^n$ or whether it is even possible to go beyond entire functions at all.

Summed up briefly, the method used to prove Theorem 9 and Theorem 10 is the following: first we introduce a certain special functional calculus, whose only purpose is to make the already opaque proofs and formulas as transparent as possible. Namely, suppose p is a polynomial of degree m and f is a smooth function. We shall consider expressions of the form

$$p(\nabla_x)f(x) := \sum_{k=0}^{m} \frac{(\nabla_x\nabla_t)^k}{k!} p(t)f(x)\Big|_{t=0},$$

as well as their "dual" counterparts

$$f(\nabla_x)p(x) := \sum_{k=0}^{m} \frac{(\nabla_x\nabla_t)^k}{k!} p(x)f(t)\Big|_{t=0},$$

meaning that

$$p(\nabla_x)f(x)\big|_{x=0} = f(\nabla_x)p(x)\big|_{x=0}. \tag{46}$$

It is then obvious that in the special case when $f(y) = e^{t\cdot y}$, the corresponding operator acts like a translation

$$e^{t\cdot\nabla_x}p(x) = p(x+t),$$

that

$$p(\nabla_x)e^{t\cdot x}f(x) = e^{t\cdot x}p(t+\nabla_x)f(x),$$

and

$$p(\nabla_t)e^{t\cdot x}\big|_{t=0} = p(x), \tag{47}$$

The equality (47) moreover suggests that to compute the Bergman projection of a polynomial p, it suffices to show that

$$(P_\alpha e^{t\cdot y})(x) := \int\limits_{\mathbb{R}^n} R_\alpha(x,y)e^{t\cdot y}\,\mathrm{d}\mu_\alpha^n(y) = e^{\frac{|t|^2}{4\alpha}} R_{\frac{1}{2}}(x,t), \tag{48}$$

whence by differentiation under the integral sign

$$\begin{aligned}
(P_\alpha p)(x) &= \int\limits_{\mathbb{R}^n} R_\alpha(x,y)p(y)\,\mathrm{d}\mu_\alpha^n(y) = \int\limits_{\mathbb{R}^n} R_\alpha(x,y)p(\nabla_t)e^{t\cdot y}\big|_{t=0}\,\mathrm{d}\mu_\alpha^n(y)\\
&= p(\nabla_t)\int\limits_{\mathbb{R}^n} R_\alpha(x,y)e^{t\cdot y}\mathrm{d}\mu_\alpha^n(y)\big|_{t=0} = p(\nabla_t)e^{\frac{|t|^2}{4\alpha}} R_{\frac{1}{2}}(x,t)\big|_{t=0},
\end{aligned}$$

or, due to (46),

$$(P_\alpha p)(x) = e^{\frac{\Delta_t}{4\alpha}} R_{\frac{1}{2}}(x,\nabla_t)p(t)\big|_{t=0}.$$

If we introduce just another piece of notation, related to the well-known Pochhammer symbol

$$(a)_k := a(a+1)(a+2)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

for which obviously

$$(a)_k = (a\tau_a)^k 1, \qquad \frac{1}{(a)_k} = \left(\frac{1}{a}\tau_a\right)^k 1 \qquad \forall k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\},$$

where τ_a is the translation operator of a by 1:

$$\tau_a f(a) := f(a+1),$$

and the corresponding polynomials thereof, defined as expressions of the form

$$p(a\tau_a x)1 = \sum_{k=0}^{m} \frac{(a)_k}{k!}(x\cdot\nabla)^k p(0), \qquad p\left(\frac{1}{a}\tau_a x\right)1 = \sum_{k=0}^{m} \frac{1}{(a)_k k!}(x\cdot\nabla)^k p(0),$$

we can in fact show the following

Lemma 1 (Harmonic Bergman projection formula)**.** *The Bergman projection of a polynomial takes the form*

$$\left(P_\alpha p\right)(x) = e_m^{\frac{\Delta_y}{4\alpha}} e_m^{x\cdot\nabla_y BC' - \frac{1}{4}|x|^2\Delta_y BC'^2} p(y)\Big|_{y,\gamma,\beta=0}, \qquad m \geq \deg p,$$

where

$$B := (b+\beta)\tau_\beta \qquad C' := \frac{1}{b+\gamma}\tau_\gamma$$

and e_m^x *is the "truncated" exponential:*

$$e_m^x := \sum_{k=0}^{m} \frac{x^k}{k!}.$$

This result is in turn used to prove the following theorem, which, along with certain results on the asymptotic behaviour of a variety of kinds of hypergeometric functions that pop up here an there (and a handful of auxiliary results that we do not mention here), is of crucial importance for the proof of Theorems 9 and 10:

Theorem 11. *Let* p_M *be a polynomial of degree* M*. If* x *and* z *are non-collinear, then the integral*

$$\int_{\mathbb{R}^n} p_M(y) R_\alpha(x,y) R_\alpha(z,y)\,\mathrm{d}\mu_\alpha^n(y) \tag{49}$$

admits the following representation:

$$\begin{aligned}
&\int_{\mathbb{R}^n} p_M(y) R_\alpha(x,y) R_\alpha(z,y)\,\mathrm{d}\mu_\alpha^n(y) \\
&\quad = \quad e^{\frac{\Delta_t}{4\alpha}} e^{x\cdot\nabla_t BC' - \frac{1}{4}|x|^2\Delta_t BC'^2} \\
&\qquad\qquad e^{\alpha z\cdot\nabla_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)B_2C_2' - \alpha^2|z|^2\left(\frac{\Delta_t}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2 + \nabla_t\cdot x\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2{C_2'}^2} p_M(t) \\
&\qquad\qquad \Phi_2\left(\begin{matrix} b+\beta \\ b+\gamma \quad b+\gamma_2 \end{matrix}; \begin{matrix} b+\beta_2 \quad b+\beta_2 \\ - \end{matrix}; \alpha u_{z,x}, \alpha\bar{u}_{z,x}\right)\Bigg|_{t,\beta,\beta_2,\gamma,\gamma_2=0}.
\end{aligned} \tag{50}$$

In case $z = \xi x$*, there is the representation*

$$\begin{aligned}
&\int_{\mathbb{R}^n} p_M(y) R_\alpha(x,y) R_\alpha(z,y)\,\mathrm{d}\mu_\alpha^n(y) \\
&\quad = e^{\frac{\Delta_t}{4\alpha}} e^{x\cdot\nabla_t BC' - \frac{1}{4}|x|^2\Delta_t BC'^2} e^{\alpha\xi x\cdot\nabla_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)B_2C_2' - \frac{1}{4}\alpha^2\xi^2|x|^2\Delta_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2 B_2{C_2'}^2} p_M(t) \\
&\qquad\qquad {}_2F_2\left(\begin{matrix} 2b+\gamma_2 & b+\beta \\ b+\gamma_2 & b+\gamma \end{matrix}; \alpha\xi\,|x|^2\right)\Bigg|_{t,\beta,\beta_2,\gamma,\gamma_2=0},
\end{aligned}$$

where $u_{z,x} = z \cdot x + \mathrm{i}\sqrt{|z|^2 |x|^2 - (z \cdot x)^2}$ *and the operators* B, B_2, C', C_2' *are defined as*

$$B = (b+\beta)\tau_\beta, \qquad B_2 = (b+\beta_2)\tau_{\beta_2}, \qquad C' = \frac{1}{b+\gamma}\tau_\gamma, \qquad C_2' = \frac{1}{b+\gamma_2}\tau_{\gamma_2}.$$

Here, Φ_2 is a slightly generalized version of the Φ_2 function introduced in (44) (and formally denoted by the same symbol), this time defined for every $x, y \in \mathbb{C}$ by the formula

$$\Phi_2\left(\begin{matrix} a \\ c_1 \quad c_2 \end{matrix}; \begin{matrix} b_1 \quad b_2 \\ - \end{matrix}; x, y\right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c_1)_{j+k}(c_2)_{j+k}} \frac{x^j y^k}{j!k!}. \tag{51}$$

For the remaining parts of the proof, we refer to [12].

8. Presentations related to the thesis

(1) Summer School Analysis – with Applications to Mathematical Physics, August 29 – September 2, 2011, Göttingen, Germany.
Talk: Harmonic Berezin Transform on the half-space.

(2) International Conference on Differential Equations, Difference Equations and Special Functions, September 3 – 7, 2012, Patras, Greece.
Talk: Harmonic Berezin Transform on the half-space.

(3) 15th International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, June 7 – 12, 2013.
Talk: Harmonic Berezin transform on various domains.

(4) Moduli Operads Dynamics I, July 9 – 11, 2013, Kongsberg, Norway.
Talk: Harmonic Berezin transform on various domains.

(5) International Conference "Mathematics Days in Sofia", July 7–10, 2014, Sofia, Bulgaria.
Talk: Harmonic Berezin transform of two arguments.

9. Publications constituting the body of the thesis

[1] J. Jahn, *On asymptotic expansion of the harmonic Berezin transform on the half-space*, J. Math. Anal. Appl. 405 (2013), 720–730.

[2] P. Blaschke, J. Jahn, *Berezin transform of two arguments.* J. Funct. Anal. 268 (2015), 3790–3833.

10. Papers citing the publications constituting the body of the thesis

[1] S. T. Ali; M. Engliš, M. (2014). *Hermite polynomials and quasi-classical asymptotics*, J. Math. Phys. 55(4), 42-102.

[2] M. Engliš, *High-power asymptotics of some weighted harmonic Bergman kernels*, submitted.

11. Other papers

[1] J. Jahn, J. Jahnová, *Towards the Proof of Yoshida's conjecture*, submitted.

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