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Report on the Habilitation Thesis
"Chaos, topological entropy and minimality in discrete dynamics"
by
Ľubomír Snoha

The theory of chaotic dynamical systems has attracted a lot of interest. In mathematics this interest is due to fascinating results and because of its applications to other areas. It should be noted, that the origins of this theory are from applications of mathematics, especially from physics. On the other hand the mathematical results obtained in this theory often have influence on applications of mathematics, for example in physics, meteorology, biology and economics. Of course also questions arised in these applications have influence on mathematics.

Roughly spoken, chaos means that a deterministic system behaves like a probabilistic system. This means, although the system is theoretically completely predictable, it behaves practically like a probabilistic system. The theory of dynamical systems splits into different areas, as continuous dynamical systems ("differential equations") and discrete dynamical systems ("difference equations"), or measure-theoretic (probabilistic) dynamical systems, topological dynamical systems and smooth dynamical systems. Ľubomír Snoha works mainly in the field of discrete topological dynamical systems. This means there is state space X (usually a compact metric space), and a (usually continuous) map $f : X \to X$ (note that in general f is not invertible). If we start at the state x at time 0, we are in state $f(x)$ at time 1, in state $f^2(x)$ at time 2, and so on. The sequence $(f^n(x))_{n=0}^{\infty}$ describing the states at time n if one starts in x is called the orbit of x under f. Chaotic behaviour means that small perturbations of the starting state x have large influence on the orbit of x. However, note that there are different "definitions" of "chaos", and they are not equivalent.

In the present habilitations thesis three different types of "chaos" are investigated, namely chaos in the sense of Li and Yorke, topological entropy (which can be considered in some sense as a measure of chaotic behaviour) and minimality (which means in some sense that a system is "slightly chaotic"). Besides of these types of "chaos" there are also other ones, for example chaos in the sense of Devaney, topological transitivity, sensitive dependence to initial conditions, distributional chaos and the Sharkovskii order. Although not presented in the habilitation thesis Ľubomír Snoha made important contributions concerning some of these other types of "chaos" (note that some of these types are dealt with in the thesis). There would have been the possibility to choose other papers to be included in the habilitation thesis, which would present the work of the author from another point of view, but of a similar quality.

This habilitation thesis contains six papers of Ľubomír Snoha published between 1990 and 2003 showing (only) a (small) part of his work. The author groups these paper into three groups each of them containing two papers, and gives a heading to each of these groups. These headings are "chaos" (meaning chaos in the sense of Li and Yorke), "topological entropy" and "minimality". For each of these groups he describes the state of research before his papers were written, the main results of his papers, and further results (by different authors) based on his papers and some open problems. These chapters make it convenient for readers, because the definitions (and previous results) necessary for reading the papers are presented, and the results of the papers are stated in a unified presentation. It is a good survey of the six papers contained in the thesis. Moreover, it is written very well.

After a bibliography containing 105 items there follow the reprints of the six papers. At this point it has to be said that the author proved a lot of important and deep results, wrote many papers of highest scientific quality, and his research had a remarkable influence on the development of dynamical systems. It is extremely unlikely that the same will not be true in the future. Ľubomír Snoha is one of the leading experts in discrete topological dynamical systems.

The question investigated in the papers summarized under the heading "chaos" is the following. In the "classical definition" of "chaos in the sense of Li and Yorke" it is required that there exists an uncountable set S with $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ for any $x \neq y \in S$. Points x, y with the property described above are called a Li-Yorke-pair. However, this definition means, that a map with a "small chaotic subset" (for example a set with Lebesgue measure 0) is chaotic in the sense of Li and Yorke, but in "reality" we will never see this chaotic behaviour. Here it should be mentioned that this phenomenon occurs for many definitions of chaos, for example it also appears defining a map as chaotic, if its topological entropy is positive. One possibility to overcome this problem is to define a map to be chaotic, if there is a "large" set of Li-Yorke-pairs. One calls f to be "generically chaotic", if the set of Li-Yorke-pairs is residual in $X \times X$, and f is said to be "densely chaotic", if the set of Li-Yorke-pairs is dense in $X \times X$. In his papers Ľubomír Snoha gave characterizations of "generic chaos" and "dense chaos" for continuous interval maps. Moreover, relations between these types of chaos, topological entropy and the Sharkovskii order are presented.

One of the two papers summarized under the heading "topological entropy" deals with topological entropy of nonautonomous systems. This means that instead of orbits $(x, f(x), f^2(x), \dots)$ considered in the classical (autonomous) theory one considers orbits $(x, f_1(x), (f_2 \circ f_1)(x), (f_3 \circ f_2 \circ f_1)(x), \dots)$ for a sequence (f_n) of continuous functions. In the above mentioned paper the topological entropy is defined for these nonautonomous systems. Some classical results are extended to nonautonomous systems. Results concerning nonautonomous systems of interval maps are strongly connected with results on skew products of interval maps (also called triangular maps on the square). One major difference to the autonomous situation is that lower semi-continuity of the topological entropy for interval maps does not hold in general. The other paper in this section investigates the question of the "minimal entropy" (in fact, it is the infimum of the topological entropy, and the question, if the minimum exists) of certain "chaotic" maps on given compact metric spaces X. In this paper chaos in the sense of Devaney is considered. For a compact metric space X one defines $I^D(X) := \inf h(f)$, where the infimum is taken over all continuous maps $f : X \to X$ which are chaotic in the sense of Devaney. It is known that there are X with $I^D(X) > 0$ and there are X with $I^D(X) = 0$, and in both cases it is possible that the minimum is attained, and that there is no minimum. For example, in the case $X = [0, 1]$, $I^D(X) = \frac{\log 2}{2}$ and the minimum is attained. In the paper it is proved that maps chaotic in the sense of Devaney can be extended to a skew product which is again chaotic in the sense of Devaney. Then $I^D(X)$ is calculated for some spaces together with the infimum of the entropy taken only over all skew products.

Finally, there are two papers summarized under the heading "minimality". Minimal systems are from a certain point of view systems which are "slightly chaotic", but not "very chaotic". The first papers investigates the question, if minimal maps are invertible. It has been known that there exist noninvertible minimal systems, but on some spaces only invertible examples of minimal maps were known. In this paper some properties of minimal maps are proved, for example that for every minimal map the image of an open set has always nonempty interior. Then it is shown that for every minimal map the set of points having only one preimage is a G_δ-set, which means that minimal maps are (in some sense) almost invertible. Moreover, examples for noninvertible minimal systems on the torus are given. The second paper deals with the stroboscopical property, this means for every strictly increasing sequence (n_k) of natural numbers and for any y there is an x such that y is a limitpoint of $(f^{n_k}(x))_{k=1}^{\infty}$. Examples of minimal homeomorphisms which do not have the stroboscopical property are given. Moreover, further results are presented relating the stroboscopical property and stronger versions of the stroboscopical property with distality, topological transitivity and mixing properties. In the paper also an example of a weakly mixing subshift of the two-sided shift on two symbols, which is chaotic in the sense of Devaney, but which does not satisfy the stroboscopical property, is given.

This habilitation thesis shows that Ľubomír Snoha made substantial contributions to the theory of discrete topological dynamical systems. The composition of the papers included in this thesis is well done, and presents the work of Ľubomír Snoha on different kinds of chaos and its influence in this area of mathematics. Mainly the important work of the author on chaos in the sense of Li and Yorke, topological entropy for nonautonomous systems, connections between topological entropy and other types of chaos, and minimal systems are shown. It has been presented in a way showing that the author worked on many different problems in this field. The disadvantage of this presentation is that between some papers contained in this thesis there are no connections. According to the scientific work of the author there would have been different possibilities to compose other papers to a thesis of a similar quality presenting his contributions from another point of view. This could have been done in a similar way as in the present one using papers on different problems (but without connections between some of them), or presenting only papers connected with one common problem (in this case not showing the broadness of the work). One can see from these remarks that Ľubomír Snoha wrote a lot of papers of highest scientific quality, and any (suitable) composition of his papers to a thesis could be criticized because of showing only a small part his work.

Before I come to the conclusion I like to make some remarks. Since I have attended some talks of Ľubomír Snoha, I know that his excellent talks are among the best ones. Because of this I am also sure that he is an excellent teacher. Concerning the extremely high scientific quality of this thesis I like to say that it is far about the average quality of a habilitation thesis, and it should not be considered as a standard for habilitation thesis.

Ľubomír Snoha made substantial contributions in the theory of dynamical systems. Some of these contributions are well presented in his habilitation thesis. Besides of the papers contained in the thesis he wrote a lot of other papers of highest scientific quality. He is one of the leading experts in discrete topological dynamical systems. Therefore I strongly recommend to promote Ľubomír Snoha to "Docent".

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