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Two counterexamples concerning integrable partial differential equations

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Contents

1. INTRODUCTION

The thesis concerns two independent papers [1] and [2]. The common subjects of both these papers are nonlinear integrable partial differential equations, their recursion operators and hierarchies of symmetries generated by them.

In the first paper an example is given of a recursion operator and its inverse having two unusal properties: Both of them are second order integro-differential operators and each of them lowers the order of symmetries in a half of hierarchy. The corresponding positive and negative hierarchies consist of local symmetries alone.

The second paper gives an example of the inverse recursion operator and a zero curvature representation, both related to nontrivial and nonsemisimple Lie algebra, for which the relationship between the zero curvature representation and nonlocal terms in the inverse recusion operator holds as well as in examples in the simple algebra case known so far.

2. SYMMETRIES AND RECURSION OPERATORS

A real-valued smooth function $F : J^n \longrightarrow \mathbb{R}$ in a p dependent variables $x = (x^1, \ldots, x^p)$, a finite number q of independent variables u^α and their derivatives u_I^α, where I denotes a finite symmetric multiindex in the independent variables, defined on an open subset of the n-th jet space, is called a *differential function* of *order* n. A function $F : J^\infty \longrightarrow \mathbb{R}$, depending on a finite number of derivatives of order at most n, is considered to be a differential function of order n.

Let $F(x, u^{(n)})$ be a differential function of order n. The *total derivative* of F with respect to x^i is the $(n + 1)$-st order differential function $D_i F$ satisfying

$$D_i F(x, j_x^{n+1} f) = \frac{\partial}{\partial x^i} F(x, j_x^n f)$$

for any smooth function $f : X \longrightarrow \mathbb{R}$. The total derivatives can be understood as vector fields on J^n.

In this and all following sections we will restrict ourself to the case of partial differential equations of two independent variables:

Consider a system of nonlinear partial differential equations

$$\Delta_\mu(x, y, u^{(n)}) = 0, \qquad \mu = 1, \ldots, l, \tag{1}$$

in two independent variables x, y, where $\Delta_\mu = \Delta_\mu(x, y, u^\alpha, u_x^\alpha, u_y^\alpha, \ldots, u_I^\alpha)$, $\alpha = 1, \ldots, q,$ are smooth differential functions of order n depending on independent and dependent variables and finite number of derivatives of the dependent variables; here I is the symmetric multiindex running over the set $\{x, y\}$.

By *equation manifold* $\mathcal{E}$ associated with the system (1) we mean the submanifold of J^∞ determined by the infinite system of equations

$$D_I \Delta_\mu = 0,$$

where I is running over all symmetric multiindices in independent variables $\{x, y\}$. Note that the total derivatives D_x, D_y are tangent to $\mathcal{E}$ so the equation itself can be identified with the equation manifold equipped by restricted total derivatives.

Consider a differential function F of x, y, dependent variables u^j and its derivatives u^j_I, $j = 1, \ldots, q$. The *directional derivative* or *linearization* of F along q-component vector $U = (U^1, \ldots, U^q)$ is given by expression

$$\ell_F[U] = \sum_{j=1}^{q} \sum_I \frac{\partial F}{\partial u^k_I} U^k_I.$$

Geometrically, the linearization $V\mathcal{E}$ can be introduced as the vertical vector bundle $V\mathcal{E} \longrightarrow \mathcal{E}$ with respect to the projection $\mathcal{E} \longrightarrow M$ on the base manifold; U^k are then additional dependent variables along the fibers of the projection $V\mathcal{E} \longrightarrow \mathcal{E}$.

Variables $x, y, u^\alpha_x, u^\alpha_y, \ldots, u^\alpha_I, \ldots$ are called *local*. Additional variables z^i, $i = 1, \ldots, k$ are called *nonlocal variables* if there are differential functions f^i, g^i depending on a finite number of local variables such that the system of equations

$$z^i_x = f^i, \quad z^i_y = g^i, \quad i = 1, \ldots, k \tag{2}$$

is fulfilled and system (2) is compatible as a consequence of (1). We do not need the more general case when f and g may depend on nonlocal variables as well.

A *morphism* of equation manifolds $\mathcal{E}, \mathcal{E}'$ is the mapping $\mathcal{E}' \longrightarrow \mathcal{E}$ which commute with projections to the base manifolds and preserves the total derivatives. A bijective morphism of equation manifolds is called an *isomorphism*. The inverse of an isomorphism is also an isomorphism. A morpism maps solutions of the the system to solutions.

A *covering* over an equation manifold $\mathcal{E}$ is a pair consisting of another equation manifold $\mathcal{E}'$ and a surjective morphism $\mathcal{E}' \longrightarrow \mathcal{E}$. Two coverings $\mathcal{E}', \mathcal{E}''$ are *isomorphic* over $\mathcal{E}$ when there exists an isomorphism $\mathcal{E}' \longrightarrow \mathcal{E}''$ that commutes with the projections to $\mathcal{E}$. A covering is *trivial* if it is isomorphic to one with $f^i = 0$, $g^i = 0$.

One can define a *symmetry* of (1) as a real-valued, vector-valued or matrix-valued differential function U that satisfy $\ell_\Delta[U] = 0$ on solution manifold $\mathcal{E}$ of (1).

Morphisms $\mathcal{E} \longrightarrow V\mathcal{E}$ that are section of the bundle $V\mathcal{E} \longrightarrow \mathcal{E}$ corresponds to symmetries of $\mathcal{E}$. A *nonlocal* symmetry corresponds to a morphism $\mathcal{E}' \longrightarrow V\mathcal{E}$ over $\mathcal{E}$, where $\mathcal{E}'$ is a covering of the original equation.

Consider the system (1) together with $2k$ equations (2). This pair generates a covering where $\mathcal{E}' = \mathcal{E} \times \mathbb{R}^k$ is a trivial vector bundle and $z^1, \ldots, z^k$ are coordinates on $\mathbb{R}^k$. If f^i, g^i are functions defined on $\mathcal{E}'$ such that vector fields

$$D'_x = D_x + \sum_{i=1}^{k} f^i \frac{\partial}{\partial z^i}, \quad D'_y = D_y + \sum_{i=1}^{k} g^i \frac{\partial}{\partial z^i}$$

commute, then $\mathcal{E}'$ with total derivatives D'_x, D'_y is k-dimensional covering over the equation manifold $\mathcal{E}$

Let $\mathfrak{g}$ be a matrix Lie algebra with associated connected and simply connected matrix Lie group G. A $\mathfrak{g}$-*valued zero-curvature representation* (ZCR) for $\mathcal{E}$ is a $\mathfrak{g}$-valued 1-form $\alpha = A dx + B dy$ defined on $\mathcal{E}$ such that

$$D_y A - D_x B + [A, B] = 0$$

holds on $\mathcal{E}$, where A, B are $\mathfrak{g}$-valued differential functions, which depends on x, y, u and its derivatives u_I and possibly on an essential (*spectral*) parameter λ.

Given an arbitrary G-valued function H we have another ZCR

$$\alpha^H = A^H dx + B^H dy,$$

where

$$A^H = D_x H H^{-1} + H A H^{-1}, \quad B^H = D_y H H^{-1} + H B H^{-1}.$$

The new α^H is called *gauge equivalent* to α. A ZCR is *trivial* when it is gauge equivalent to zero. In such a case $A = D_x H H^{-1}$, $B = D_y H H^{-1}$. A covering $\mathcal{E}' \longrightarrow \mathcal{E}$ is said to *trivialize* a ZCR α if the pullback of α along the morphism $\mathcal{E}' \longrightarrow \mathcal{E}$ is a trivial ZCR.

In Olver's formalism (see [Ol1] for the details), a recursion operator is a linear integro-differential (pseudodifferential) operator, which maps symmetries to symmetries. Integro-differential operators involve inverses D^{-1} of the total derivative operator $D = D_x$. Here D^{-1} is formally defined by identities $D \circ D^{-1} = \mathrm{id}$, $D^{-1} \circ D = \mathrm{id}$. However, the latter identity is actually invalid. To fix such a weakness, G. A. Guthrie [G] introduced another definition of recursion operator. It was equivalently formulated in [M] as follows: A *recursion operator* for an equation manifold $\mathcal{E}$ is a pair of coverings $K, L : R \longrightarrow V\mathcal{E}$ over the linearization $V\mathcal{E}$ of $\mathcal{E}$ such that K and L commutes with projections of $V\mathcal{E}$ to $\mathcal{E}$.

Given a recursion operator $\mathcal{R}$ for a system of differential equations, we can generate an infinite hierarchy of local symmetries from any one symmetry U by repetitive applying $\mathcal{R}$ to U [Ol2]. Since the operator $\mathcal{R}$ is usually a differential operator of order greater than 1, such a hierarchy consits of increasing-order symmetries.

According to J. M. Verosky [V], the hierarchy of symmetries generated by a recursion operator may be extended to the negative direction by inverting the operator $\mathcal{R}$. The inverted recursion operator $\mathcal{R}^{-1}$ generates an analogic hierarchy of symmetries, which are, as a rule, nonlocal. We call this half of the hierarchy a *negative* part in opposite to the original Olver's part called *positive*.

3. Can we always distinguish between positive and negative hierarchies?

In the paper [1] we consider the partial differential equation

$$u_{xy} = u u_{xx} + \tfrac{1}{2}\, u_x^2 + u, \tag{3}$$

which is a particular case of the transformed Manna–Neveu generalization [MN] of the Hunter–Saxton equation ([HS] and [HZ]).

We give two recursion operators $\mathfrak{N}$ and $\mathfrak{R}$ for this equation. Namely, we observe that the right-hand side of the evolution equation

$$u_t = \frac{u_{xxx}}{(2\, u_{xx} + 1)^{3/2}},$$

happens to be a symmetry for (3). This evolution equation can be transformed into the Harry Dym equation, and this yields the first recursion operator $\mathfrak{N}$ for both (3) and the evolution equation in question. The second recursion operator $\mathfrak{R}$ originates from the zero curvature representation for (3) through the procedure suggested in [MSe].

The operators $\mathfrak{R}$ and $\mathfrak{N}$ are mutually inverse second order integro-differential operators. Each of them increases the order of symmetries in a half of hierarchy and lowers the order in the other half. *Both* the positive and negative hierarchies consist of *local* symmetries. So, in this particular example, there is no difference between the positive and negative hierarchy.

4. A conjecture concerning nonlocal terms of recursion operators

In the paper [2] we investigate a relationship between the zero curvature representation of nonlinear partial differential equation in dimension two and the algebra of recursion operators. As observed in [MSe], if $\mathfrak{R}$ is a ”conventional” recursion operator of an integrable system, then the nonlocal variable Ψ of $\mathfrak{N}$, $\mathfrak{N} = (\mathfrak{R} + \lambda)^{-1}$ turns out to be $\mathfrak{g}$-valued and related to zero-curvature representation $Adx + Bdy$ in a unique way

$$\begin{aligned} \Psi_x &= [A, \Psi] + \ell_A[U], \\ \Psi_y &= [B, \Psi] + \ell_B[U], \end{aligned} \tag{4}$$

where U denotes an arbitrary symmetry. The parameter λ can be identified with the spectral parameter of the zero curvature representation.

In the examples considered so far the Lie algebra $\mathfrak{g}$ was a simple algebra (or an abelian algebra). However, in the general case of $\mathfrak{g}$ having a nontrivial solvable part examples seemed to suggest that (4) is invalid.

The simplest counterexample was the inverse recursion operator for the mKdV equation, considered in [G] and [M]. In [2] this example is revised and shown that (4) holds.

Then we consider another equation considered by M. V. Foursov in [F] together with its direct and inverse recursion operator, both being related to a non-trivial nonsemisimple algebra. It is shown that (4) is also fulfilled in this case.

These facts suggest universal validity of (4), but further examples are to be considered.

PUBLICATIONS CONCERNING THE THESIS

[1] Baran H, Can we always distinguish between positive and negative hierarchies?, *J. Phys. A: Math. Gen.* **38** (2005) L301–L306

[2] Baran H, Marvan M, A conjecture concerning nonlocal terms of recursion operators *Fundamental'naya i Prikladnaya Matematika*, to appear

PRESENTATIONS

[3] Seminar on differential geometry, Università degli Studi di Lecce, Lecce, Italy, November 24–29, 2004. Invitation. Talk on: "An unusual recursion operator for a generalized Hunter–Saxton equation," "Computer algebra and zero curvature representations computation method".

[4] The 25th Winter School Geometry and Physics, Srní, Czech Republic, January 15–22, 2005. Talk on: "An unusual recursion operator for a generalized Hunter–Saxton equation".

[5] Geometry in Odessa — 2005. Differential Geometry and its Applications, Odessa, Ukraine, May 23–29, 2005. Talk on: "Can we always distinguish between positive and negative hierarchies?"

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