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On dynamical behaviour of the aggregate macroeconomic equilibrium models

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1. Introduction

This thesis is based on three independent papers [P1], [P2] and [P3]. The common subject is the dynamical behaviour of the macroeconomic equilibrium models. This abstract consists of three parts. The first part is focused on selected kinds of dynamics in the plane $\mathbb{R}^2$, namely local stability analysis, relaxation oscillations and chaos. The second part presents an overview of macroeconomic equilibrium models used in this thesis; the starting point upon which these models are built is the fundamental macroeconomic model called the IS-LM model. In the third part the dynamical behaviour of these models is described.

The first paper [P1] deals 1) with definitions and descriptions of particular functions used in the original IS-LM model equipped by the so-called Kaldor's condition, and 2) with the stability of the model based on these functions. In the second paper [P2] the sufficient condition for the existence of relaxation oscillations in the original IS-LM model is formulated and proved. The third paper [P3] is focused on the chaos existence in a dynamical system generated by the Euler equation branching in the plane $\mathbb{R}^2$, i.e. a special type of differential inclusion in the plane $\mathbb{R}^2$, and its application to a new macroeconomic equilibrium model called the IS-LM/QY-ML model.

2. Description of the dynamics in the Euclidean plane

Many models describing dynamics in economies are given by two dimensional autonomous systems in the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned} \tag{1}$$

where f_1 and f_2 are non-linear real functions, see e.g. [8], [15], [25]. Such a system defines a *continuous dynamical system* in the Euclidean plane. The functions f_1 and f_2 are usually assumed to be continuous and differentiable. The standard analysis of dynamical behaviour of the systems (1) lies in the local stability analysis, i.e. in the classification of singular points using eigenvalues of Jacobi's matrix of the linearised system to (1), see e.g. [1], [9], [14] or [24], where the singular points are defined to be the intersection points of the curves given by the equations $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$. In this thesis, we focus on the hyperbolic singular points, i.e. stable nodes (negative real eigenvalues), stable foci (complex eigenvalues with negative real part), unstable nodes (positive real eigenvalues), unstable foci (complex eigenvalues with positive real part) and saddles (one positive and one negative real eigenvalue).

Another point of view on the dynamics is hidden in the existence of different types of oscillations in these systems. We focus on the so-called *relaxation oscillations*, which are a type of oscillation resembling a limit cycle, see e.g [6], [9] or [12]. The relaxation oscillations emerge in the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= \varepsilon f_2(x_1, x_2), \end{aligned} \tag{2}$$

where ε is a small positive parameter. Here, the variable x_2 changes very slowly during the time compared with the variable x_1. So, we can consider the system $\dot{x}_1 = f_1(x_1, x_2), \dot{x}_2 = 0$ instead of the system (2). Thus, only the equation $\dot{x}_1 = f_1(x_1, x_2)$ remains to be considered where the variable x_2 stands for a parameter. It follows that all points lying on the curve given by the equation $f_1(x_1, x_2) = 0$ are singular. Then, we can define a stable and an unstable arc to be the set of all possible stable and unstable singular points of the system (2) lying on this curve, respectively. If the functions f_1 and f_2 are such that there is one unstable regime located between any two stable regimes, then there exist relaxation oscillations in this system, see e.g [6], [9] or [12]. An example illustrating a typical vector field given by the right-hand side of such a system is shown in Figure 1. The red cycle in Figure 1 represents

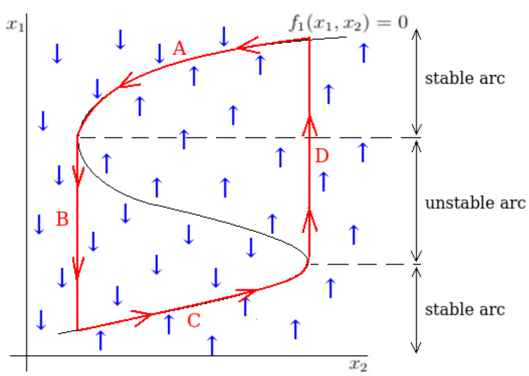

FIGURE 1. Relaxation oscillations

the relaxation oscillations in this system. Almost all trajectories of this system are directed upwards or downwards, i.e. are oriented in the direction from unstable to stable arcs, see the blue arrows in Figure 1 indicating the direction of the trajectories, and the velocity of the motion of the moving point is infinitely large on these vertical trajectories. Only the moving point located on or near the stable arcs has a finite velocity of motion and goes on or along the stable arcs. So, the cycle consists of four parts - two parts with an infinitely large velocity of motion (see part B and D) and two parts along the stable arcs with a finite velocity of motion (see part A and C).

It is a known fact that in the two-dimensional systems of the form (1) no chaos arises, see [9], [14], [15], [25]. However, in the last decade a new approach to the study of two-dimensional dynamics based on the continuous dynamical system generated by a special type of differential inclusion called the Euler equation branching has been developed. In [16], [21] several particular results concerning the onset of chaos in the continuous dynamical system generated by the Euler equation branching were given. In this thesis we continue this work and we present certain new results in this area.

Definition 2.1. (see [18]) The differential inclusion is given by

$$\dot{x} \in F(x), \tag{3}$$

where F is a set-valued map which associates a set $F(x) \subset \mathbb{R}^n$ to every point $x \in \mathbb{R}^n$.

Note that any autonomous system $\dot{x} = f(x)$, where f is a real function in the Euclidean n-dimensional space, can be described by a differential inclusion (3) with $F(x) = \{f(x)\}$.

Definition 2.2. (see [21]) Let $X \subset \mathbb{R}^2$ be an open set and $f, g : X \to \mathbb{R}^2$ be continuous functions. Let us consider the following differential inclusion

$$\dot{x} \in \{f(x), g(x)\}. \tag{4}$$

We say that there is the *Euler equation branching in the point* $x \in X$ if $f(x) \neq g(x)$. If there is the Euler equation branching in every point $x \in X$, then we say that there is the *Euler equation branching on the set* X.

Note that one branch of the differential inclusion (4) is the system (1).

Let $X \subseteq \mathbb{R}^2$ be a non-empty open set equipped with the Euclidean metric d and $T := [0, \infty)$ be a time index, let $F : X \to 2^{\mathbb{R}^2}$ be a set-valued function given by $F(x) := \{f(x), g(x)\}$, where $f, g : X \to \mathbb{R}^2$ are continuous functions such that the condition $f(x) \neq g(x)$ is satisfied for all $x \in X$. Let Z denote the set $\{\gamma | \gamma : T \to X\}$, where $\gamma : T \to X$ are functions that are continuous and continuously differentiable almost everywhere on T.

Definition 2.3. (see [21]) Let $F : X \to 2^{\mathbb{R}^2}$ be a set-valued function specified above. A *dynamical system generated by* F is defined to be a set

$$D := \{\gamma \in Z | \dot{\gamma}(t) \in F(\gamma(t)) \ a.e.\} \tag{5}$$

The functions γ from Definition 2.3 define the solutions of the differential inclusion (4). Note that the solutions of the Euler equation branching are the solutions of the first branch $\dot{x} = f(x)$, the solutions of the second branch $\dot{x} = g(x)$ and the switching solutions between these two branches.

Definition 2.4. (see [21]) Let $F : X \to 2^{\mathbb{R}^2}$ be a set-valued function specified above. We say that a non-empty set $V \subset \mathbb{R}^2$ is a *compact F-invariant set*, if V is compact and for each $x \in V$ there exists $\gamma \in D$ such that $\gamma(0) = x$ and $\gamma(t) \in V$ for all $t \in T$.

Definition 2.5. (see [21]) Let $a, b \in X \subseteq \mathbb{R}^2$ and D be a dynamical system in the sense mentioned above. Let $\gamma \in D$ and $t_0, t_1 \in T$ be such that $t_0 < t_1$. A *simple path from a to b generated by D* is defined to be the set $\{\gamma(t) : t_0 \leq t \leq t_1\}$ where $\gamma(t_0) = a$, $\gamma(t_1) = b$ and $\dot{\gamma}$ has only finitely many discontinuities on $[t_0, t_1]$ and $a \neq \gamma(s) \neq b$ for all $t_0 < s < t_1$.

Definition 2.6. (see [21]) Let $F : X \to 2^{\mathbb{R}^2}$ be a set-valued function specified above. Let $V \subset X \subseteq \mathbb{R}^2$ be a non-empty compact F-invariant set. Let $V^* = \{\gamma \in D | \gamma(t) \in V, \text{ for all } t \in T\}$ where $V \subset \mathbb{R}^2$ is a compact F-invariant set. The set V is called a *chaotic set* provided that

(1) for all $a, b \in V$, there exists a simple path from a to b generated by V^*,
(2) there exists $U \subset V$ non-empty and open (relative to V) and $\gamma \in V^*$ such that $\gamma(t) \in V \setminus U$ for all $t \in T$ (i.e. there exists $\gamma \in V^*$ such that $\{\gamma(t) : t \in T\}$ is not dense in V).

Stockman and Raines [21] proved that the existence of a chaotic set V implies the existence of Devaney chaos, and that the existence of a chaotic set V with a non-empty interior or homeomorphic to $[0, 1]$ where functions f and g fulfil the conditions $\|f(x)\|\|g(x)\| > 0$ and $\cos^{-1}\left(\frac{f(x)\cdot g(x)}{\|f(x)\|\|g(x)\|}\right) = \pi$ for all $x \in V$ implies the existence of Li-Yorke and distributional chaos [1]. The paper [P3] continues the work done in [21] and describes and illustrates chaotic sets $V \subset \mathbb{R}^2$ and associated chaotic sets of solutions V^*.

Theorem 2.1 and Theorem 2.2 show that any continuous dynamical system generated by the Euler equation branching (4) where the singular point x^* of the first branch f is hyperbolic and the solution of the second branch g is unbounded in $\bar{B}_\delta(x^*)$, where $\delta > 0$ is such that $g(x) \neq 0$ for every $x \in \bar{B}_\delta(x^*)$, admits the existence of chaotic sets. From this it follows that if the first branch has a hyperbolic singular point x^* and the second branch has a hyporbolic singular point y^*, and $x^* \neq y^*$, then the trajectories corresponding to both the branches are located and directed in such a way that they can give rise to a chaotic set V in the corresponding phase portrait in $\mathbb{R}^2$. In [P3], we illustrate all such cases. Note that from Definition 2.6 it follows that a sufficient condition for the existence of a chaotic set V, in addition to the "right" direction and location of the trajectories, is the existence of the set of solutions V^* associated to the chaotic set V (in the sense of Definition 2.6), i.e. the existence of an appropriate set of solutions with an appropriate branch switching. Such a set of solutions V^* is constructed in the proof of Theorem 2.3 in [P3].

Theorem 2.1. *(see* [21]*) Let $F : X \to 2^{\mathbb{R}^2}$ be a set-valued function specified above. Let $x^* \in X \subseteq \mathbb{R}^2$, $f(x^*) = 0$ and $g(x^*) \neq 0$, let $\lambda_1,\ \lambda_2$ be the eigenvalues of Jacobi's matrix of the system $\dot{x} = f(x)$ in the point x^* and $e_1,\ e_2$ be the corresponding eigenvectors. We choose $\delta > 0$ such that $g(x) \neq 0$ for every $x \in \bar{B}_\delta(x^*)$. Let the solution of $\dot{x} = g(x)$ be unbounded in $\bar{B}_\delta(x^*)$. Then, the following assertions hold.*

(1) *If there exists $\varepsilon > 0$ such that x^* is a source or a sink [2] for f on $B_\varepsilon(x^*)$, then F admits a chaotic set.*

[1]The Devaney, Li-Yorke and distributional chaoses are defined in the usual way, see [21].

[2]By a source we mean an unstable node or an unstable focus. By a sink we mean a stable node or a stable focus.

(2) *If* $\lambda_1,\ \lambda_2$ *are such that* $\lambda_1 < 0,\ \lambda_2 > 0$ *and* $g(x^*) \neq \alpha e_1$ *and* $g(x^*) \neq \beta e_2$*, where* $\alpha, \beta \in \mathbb{R} \setminus \{0\}$*, then* F *admits a chaotic set with a non-empty interior.*

Our main contribution to this area is formulated in the following two theorems and a few remarks. Note that in Theorem 2.1 the case when $g(x^*) = \alpha e_1$ or $g(x^*) = \beta e_2$, where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, is not covered. Under the assumptions of Theorem 2.1 we prove that also in this missing case, similar results are valid.

Theorem 2.2. *(see* [P3]*) Let* $\lambda_1 < 0,\ \lambda_2 > 0$ *(i.e.* x^* *is a saddle point) and* $g(x^*) = \alpha e_1$ *or* $g(x^*) = \beta e_2$*, where* $\alpha, \beta \in \mathbb{R} \setminus \{0\}$*. Then* F *admits a chaotic set.*

An example of a chaotic set V associated to the Euler equation branching ($\dot{x} \in f(x), g(x)$) is shown in Figure 2. The black trajectories correspond to the first branch ($\dot{x} = f(x)$) and the blue trajectories correspond to the second branch ($\dot{x} = g(x)$). The black trajectories illustrate the behaviour of the dynamical system in a neighbourhood of a saddle and the blue trajectories illustrate the behaviour of the dynamical system in a neighbourhood of an unstable focus. The red areas represent the chaotic sets, three ones with a non-empty interior and one homeomorphic to the unit interval.

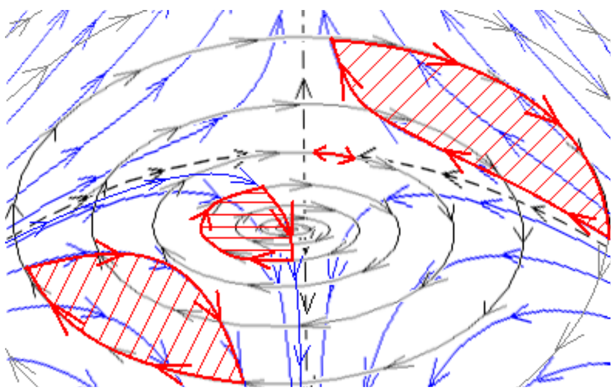

FIGURE 2. An example of a chaotic set V (see [P3])

In the following theorem we assume that the branches f and g are such that the chaotic set V is admitted, e.g. both the branches have hyperbolic singular points, say x^* corresponding to f and y^* corresponding to g, and $f(x^*) = 0,\ g(x^*) \neq 0$ and $f(y^*) \neq 0,\ g(y^*) = 0$, i.e. $x^* \neq y^*$.

Theorem 2.3. *(see* [P3]*) In a dynamical system* D *generated by the Euler equation branching in* $\mathbb{R}^2$ *there exists a chaotic set* V *(with associated set of solutions* V^**). Hence, the associated set of solutions* V^* *is Devaney, Li-Yorke and distributionally chaotic.*

Remark 2.1. The proof of Theorem 2.3 lies in the construction of an appropriate set of solutions V^*. For one set $V \subset \mathbb{R}^2$ (understood as non-empty compact subset of $\mathbb{R}^2$) there can exist uncountably many different possible sets V^* differing just in the branch switching (i.e. in the set of solutions V^*) depending on the character of the modelled problem (i.e. of the particular dynamical system). The "force" causing the switch is exogenously determined, and once more depends on the particular modelled problem and its interpretation, see the application part of this problem in Section 4.

Remark 2.2. (see [P3]) Note that if the set V^* associated to a set $V \subset \mathbb{R}^2$ with respect to a set-valued function F specified above consists of only one solution γ, then the conditions (1) and (2) from Definition 2.6 cannot be fulfilled simultaneously. Therefore, a set V with a one point set of solutions V^* associated to V is a trivial example of a non-chaotic set.

3. Aggregate macroeconomic equilibrium models

In this section we present an overview of aggregate macroeconomic equilibrium models considered in papers [P1], [P2] and [P3]. To this end, we employ both the system of differential equations (1) and a special type of differential inclusion (4) mentioned above to describe certain macroeconomic situations. The starting point for creating of these models is the fundamental macroeconomic equilibrium model called IS-LM model. In this thesis, the *aggregate macroeconomic equilibrium* is understood to be the simultaneous goods market and money (or financial assets) market equilibrium.

Definition 3.1. (see [8], [10]) The *original dynamic IS-LM model* is given by a system of differential equations

$$\begin{aligned} \text{IS:} \quad & \tfrac{dY}{dt} = \alpha[I(Y,R) - S(Y,R)] \\ \text{LM:} \quad & \tfrac{dR}{dt} = \beta[L(Y,R) - M_{CB}], \end{aligned} \tag{6}$$

where Y is the aggregate income (GDP, GNP), R is the interest rate, $I(Y,R)$ is an investment function, $S(Y,R)$ is a saving function, $L(Y,R)$ is a money demand function, $M_{CB} > 0$ is the money stock and $\alpha, \beta > 0$ are certain parameters of dynamics.

In the IS-LM model, the investment and saving functions represent the goods market, and the money demand function and the money stock represent the money market. The goods market is described by two sectors of economy: 1) households represented by the savings, and 2) firms represented by the investment. The constant money stock assumption represents an exogenous conception of the money supply. The money stock is determined by the central bank. In the original IS-LM model, the price level is assumed to be constant, so the types of the interest rate are not distinguished. The original model is demand-oriented, i.e. the supply side is fully adapted to the demand side. In summary (based on the mentioned assumptions), the original model is applicable in a state of a recessionary gap. The equality of the investment and the savings represents the goods market equilibrium. The equality of the money demand and the money supply is the money market equilibrium.

The second considered aggregate macroeconomic equilibrium model is a modified IS-LM model. In order to extend the applicability of the IS-LM model, we modify two of the assumptions made in the original model: 1) the assumption of the constant price level, 2) the assumption of the exogenous money supply. More explicitly, we suppose 1) a floating instead of a constant price level in place of the assumption 1. It requires two types of interest rates - the long-term real interest rate R on the goods market and the short-term nominal interest rate [3] i on the money market (or financial assets market) to be distinguished. The relationship between these two types of interest rates is given by $i = R - MP + \pi^e$, where MP is a maturity premium and [4] π^e is an expected inflation rate, see [4]. In the following, we assume that MP and π^e are constants, hence $\frac{di}{dt} = \frac{d(R-MP+\pi^e)}{dt} = \frac{dR}{dt}$. Secondly, we suppose 2) that the assumption of the exogenous money supply can be replaced with the assumption of the joint exogenous and endogenous money supply taking place simultaneously. Here, the endogenous money supply means that money is generated in economies by credit creation, see [3], [19]. The fact that both the endogenous and exogenous money supply can arise in the real world is illustrated e.g. by [2], [17]. Thus, the supply of money is now given by $M(Y, R - MP + \pi^e) + M_{CB}$, see [P3], where $M(Y, R - MP + \pi^e)$ is a newly defined money supply function representing the endogenous part of the money supply, and M_{CB} is the money stock (determined by the central bank) representing the exogenous part of the money supply. The modified IS-LM model is still only demand-oriented, i.e. this model describes the macroeconomic situation during one of the economic cycle phases called recession.

Definition 3.2. (see [P3], [O4]) The *modified IS-LM model* is given by the following system

$$\begin{aligned} \text{IS:} \quad & \frac{dY}{dt} = \alpha[I(Y,R) - S(Y,R)] \\ \text{LM:} \quad & \frac{dR}{dt} = \beta[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_{CB}], \end{aligned} \tag{7}$$

where Y is the aggregate income (GDP, GNP), R is the long-term real interest rate, $I(Y,R)$ is an investment function, $S(Y,R)$ is a saving function, $L(Y, R - MP + \pi^e)$ is a money demand function, $M(Y, R - MP + \pi^e)$ is a money supply function, $M_{CB} > 0$ is the money stock and $\alpha, \beta > 0$ are certain parameters of dynamics.

The last considered restrictive assumption in the original model is the demand orientation. So, thirdly, we eliminate this assumption. We create a new supply-oriented aggregate macroeconomic equilibrium model called the QY-ML model. Here, the supply-oriented model is a model that describes the economic state when the demand fully adapts to the supply. The QY-ML model describes the macroeconomic situation during the economic cycle phase called expansion. The floating price level and the money supply conception are modelled in the same way as in the modified IS-LM model.

[3] i does not denote $\sqrt{-1}$ in this thesis, i abbreviates the word "interest".

[4] π^e is the standard notation used in economy. Here, π does not mean the Ludolphian number and e does not mean the Euler number. π denotes an inflation rate and the superscript e means "expected".

Definition 3.3. (see [P3]) The *QY-ML model* is given by the following system of differential equations

$$\begin{array}{ll} \text{QY:} & \frac{dY}{dt} = \alpha[Q(\mathcal{K}(Y,R),\mathcal{L}(Y,R),\mathcal{T}(Y,R)) - Y] \\ \text{ML:} & \frac{dR}{dt} = \beta[M(Y,R-MP+\pi^e) + M_{CB} - L(Y,R-MP+\pi^e)], \end{array} \tag{8}$$

where Y is the aggregate income (GDP, GNP), R is the long-term real interest rate, $Q(\mathcal{K},\mathcal{L},\mathcal{T})$ is a production function, $\mathcal{K}(Y,R)$ is a capital function, $\mathcal{L}(Y,R)$ is a labour function, $\mathcal{T}(Y,R)$ is a technical progress function, $M(Y,R-MP+\pi^e)$ is a money supply function, $M_{CB} > 0$ is the money stock determined by the central bank, $L(Y,R-MP+\pi^e)$ is a money demand function, $MP > 0$ is a maturity premium, $\pi^e > 0$ is an expected inflation rate and $\alpha,\beta > 0$ are certain parameters of dynamics.

The modified IS-LM model is valid during the recession, the new QY-ML model is valid during the expansion. We join these two models using the Euler equation branching, see Definition 2.2, and we create a new model which describes the macroeconomic situation during all phases of the economic cycle (the recession, the trough, the expansion and the peak), called IS-LM/QY-ML model. In the peak, the new QY-ML model is switched into the modified IS-LM model, and in the trough it goes the other way round.

Definition 3.4. (see [P3]) The *overall macroeconomic IS-LM/QY-ML model* is given by the differential inclusion

$$\begin{pmatrix} \dot{Y} \\ \dot{R} \end{pmatrix} \in \left\{ \begin{pmatrix} \alpha_d[I(Y,R) - S(Y,R)] \\ \beta_d[L(Y,i) - M(Y,i) - M_{CB} \end{pmatrix}, \begin{pmatrix} \alpha_s[Q(\mathcal{K}(Y,R),\mathcal{L}(Y,R),\mathcal{T}(Y,R)) - Y] \\ \beta_s[M(Y,i) + M_{CB} - L(Y,i)] \end{pmatrix} \right\} \tag{9}$$

where $i = R - MP + \pi^e$, $M_{CB} > 0$ and $\alpha_d > 0$, $\alpha_s > 0$, $\beta_d > 0$, $\beta_s > 0$ are certain parameters of dynamics.

For the economic reasons, the investment, the saving, and the money demand functions are supposed to satisfy the following properties (see [8])

$$0 < \frac{\partial I}{\partial Y} < 1, \frac{\partial I}{\partial R} < 0, 0 < \frac{\partial S}{\partial Y} < 1, \frac{\partial S}{\partial R} > 0, \tag{10}$$

$$\frac{\partial L}{\partial Y} > 0, \frac{\partial L}{\partial R} < 0. \tag{11}$$

In addition, the so-called *Kaldor's condition* is often assumed (see [5], [11])

$$\begin{array}{lll} \frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y} & \text{for} & Y \in [0,X), \\ \frac{\partial I}{\partial Y} > \frac{\partial S}{\partial Y} & \text{for} & Y \in (X,Z), \\ \frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y} & \text{for} & Y \in (Z,\infty), \end{array} \tag{12}$$

where $\frac{\partial I}{\partial Y}$ and $\frac{\partial S}{\partial Y}$ are equal in the points X and Z $(X < Z)$ for some fixed R. This condition describes the sigma-shaped graphs of the functions $I(Y)$ and $S(Y)$ for some fixed R in the way displayed in Figure 3. We suppose that the properties of the newly defined money supply,

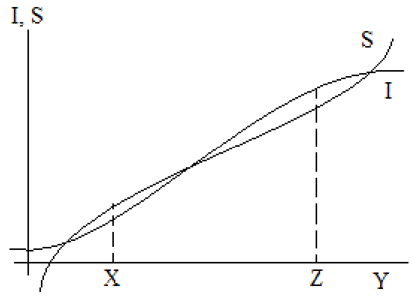

Figure 3. An illustration of Kaldor's condition (see [P1], [P2])

production function, capital function, labour function, and technical progress function are determined in the following way (see [P3])

$$0 < \frac{\partial M}{\partial Y} < \frac{\partial L}{\partial Y}, \frac{\partial M}{\partial R} > 0. \tag{13}$$

$$\frac{\partial Q}{\partial \mathcal{K}} > 0, \frac{\partial Q}{\partial \mathcal{L}} > 0, \frac{\partial Q}{\partial \mathcal{T}} > 0, \tag{14}$$

$$\frac{\partial \mathcal{K}}{\partial Y} > 0, \frac{\partial \mathcal{L}}{\partial Y} > 0, \frac{\partial \mathcal{T}}{\partial Y} > 0, \frac{\partial \mathcal{K}}{\partial R} < 0, \frac{\partial \mathcal{L}}{\partial R} < 0, \frac{\partial \mathcal{T}}{\partial R} < 0, \tag{15}$$

We also assume MP and π^e to be constants, whence $\frac{\partial L(Y,i)}{\partial i} = \frac{\partial L(Y,R-MP+\pi^e)}{\partial R}$ and $\frac{\partial M(Y,i)}{\partial i} = \frac{\partial M(Y,R-MP+\pi^e)}{\partial R}$.

The graphical representations of the models are given by the IS and LM curves, or QY and ML curves and the corresponding phase portraits (or by a sketch of the vector field given by the right-hand side of the system: every little arrow represents a vector at a given point). The IS curve is the curve defined by the equation $I(Y, R) = S(Y, R)$ and the QY curve is the curve defined by the equation $Q(\mathcal{K}(Y, R), \mathcal{L}(Y, R), \mathcal{T}(Y, R)) = Y$. The LM curve is the curve defined by the equation $L(Y, R) = M_{CB}$ for the original IS-LM model and by the equation $L(Y, R - MP + \pi^e) = M(Y, R - MP + \pi^e) + M_{CB}$ for the modified IS-LM model. The ML curve is identical to the LM curve for the modified model.

4. Dynamical behaviour of the models

As mentioned above, all considered models are based on the original IS-LM model (6). The IS-LM model was studied in many papers, see for example its continuous two-dimensional versions in [8], [23] and [22], continuous three-dimensional versions in [7], [13], [20], [26] and continuous four-dimensional version in [27]. In this section, we briefly present the results from papers [P1], [P2], [P3] and a few supplements from the papers [O3] and [O4] concerning the application of the general results on dynamical systems described in Section 2 to the aggregate macroeconomic models presented in Section 3.

In the paper [P1], we find particular investment, saving and money demand functions fulfilling conditions (10), (11) and Kaldor's condition (12). Based on these functions we create the augmented IS-LM model and study its equilibria and their stability properties.

We illustrate the macroeconomic situation described by this model on several illustrative examples, see e.g. Figure 4, and we create non-linear regression models corresponding to the particular functions. It can be proved that in the augmented IS-LM model based on the mentioned particular functions typically three singular points exist, and the first and the third one is a stable node or a stable focus and the second one (located in the middle) is an unstable saddle point as it is illustrated in Figure 4, see [P1]. In Figure 4, we can see an

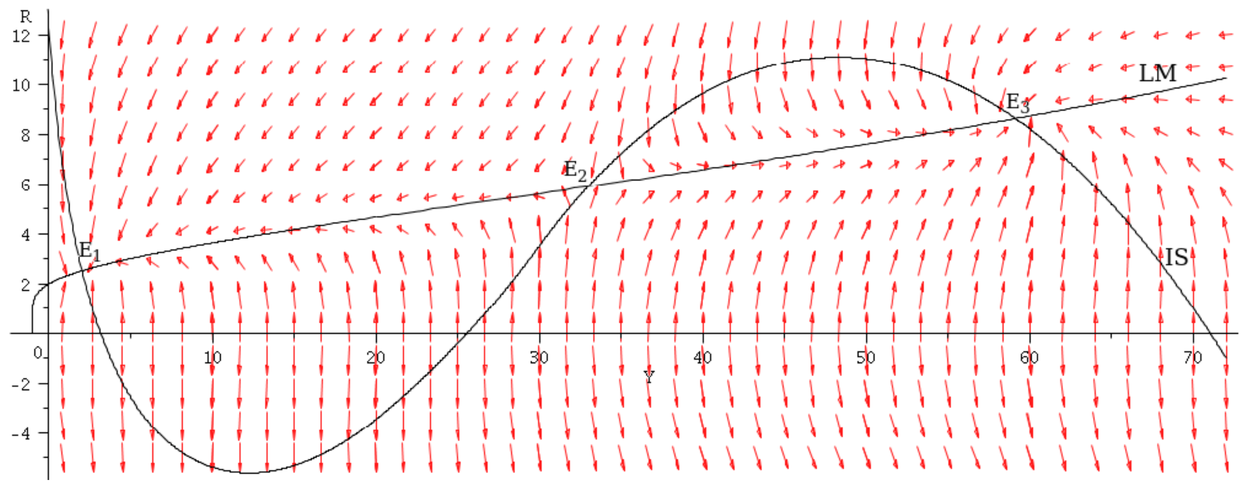

FIGURE 4. An illustrative example of the augmented IS-LM model based on particular functions (see [P1])

illustration of a behaviour of a particular dynamical system with three singular points E_1, E_2 and E_3 where the point E_1 is a stable node, the point E_3 is a stable focus and the point E_2 is an unstable saddle point.

In the paper [P2], we establish and prove a sufficient condition for the existence of the relaxation oscillations in the original IS-LM model (6). We assume that the adjustment speed of the money market is very slow compared with the adjustment speed of the goods market, i.e. the adjustment speed of the aggregate income Y is faster than the adjustment speed of the long-term real interest rate R. We describe this situation by the system (see [P2])

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,R) - S(Y,R)] \\ \frac{dR}{dt} &= \varepsilon\beta[L(Y,R) - M_{CB}], \end{aligned} \tag{16}$$

where ε is a small positive parameter. Thus, there remains only the IS curve to be treated.

Theorem 4.1. *(see* [P2]*) Let us consider an IS-LM model with very slow changes of interest rate R in the time (16) subjected to the economic properties (10), (11) and Kaldor's conditions (12). Then in such a model the clockwise relaxation oscillations arise.*

The relaxation oscillations from the previous theorem represent the relaxation oscillations on the goods market, i.e. on the IS side of the IS-LM model, and can be seen in Figure 5. Here, the arcs A_1 and A_3 are stable arcs and the arc A_2 is an unstable arc. The relaxation

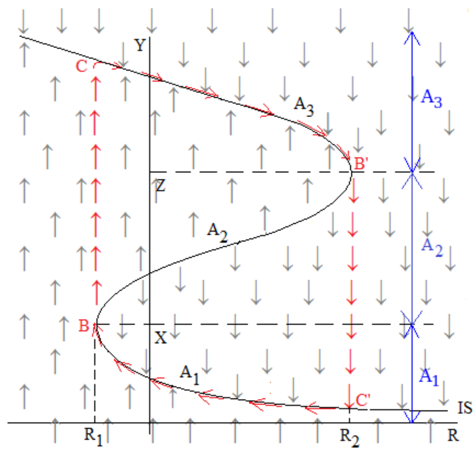

FIGURE 5. Relaxation oscillations on the goods market (see [P2])

oscillations on the goods market cause seemingly unexpected fluctuations of the aggregate income Y (see the vertical segment between the point B and C or between B' and C' in Figure 5), and the moving point has an infinitely large velocity of motion here. The relaxation oscillations (i.e. fluctuations of aggregate income Y) whose existence is guaranteed by Theorem 4.1 emerge as a consequence of the monetary policy. An analogue to Theorem 4.1 is true also for the modified IS-LM model, see [O3]. It is necessary to remark that the assumption of the adjustment speed of the aggregate income Y faster than the adjustment speed of the long-term real interest rate R is unusual.

Usually, an opposite case is assumed, i.e. that the adjustment speed of the money market is faster than the adjustment speed of the goods market, i.e. the adjustment speed of the long-term real interest rate R is faster than the adjustment speed of the aggregate income Y. We focus on this problem in [O4] for the modified IS-LM model (7). We describe this macroeconomic situation by the system (see [O4])

$$\begin{aligned} \frac{dY}{dt} &= \epsilon\alpha[I(Y,R) - S(Y,R)] \\ \frac{dR}{dt} &= \beta[L(Y, R - MP + \pi^e) - M(Y, R - MP + \pi^e) - M_{CB}] \end{aligned} \tag{17}$$

where ϵ is a small positive parameter. Thus, again only the LM curve remains to be dealt with. In [O4], we formulate a sufficient condition for the existence of the relaxation oscillations in the system (17) (provided the changes of long-term real interest rate is faster than that of the aggregate income). We call this sufficient condition the *three phases money demand and money supply* depending on the short-term nominal interest rate i (or on the long-term real interest rate R) for some fixed aggregate income Y. As the name of this sufficient condition suggests, the courses of the functions $L(i)$ and $M(i)$ are divided into three phases: 1) $i \in [0, P)$, 2) $i \in (P, Q)$ and 3) $i \in (Q, \infty)$, where $0 < P < Q$. In the first and in the third phases (i.e. for $i \in [O, P) \cup (Q, \infty)$), the money demand and the money supply behave usually as it is described in (11) and (13), i.e. $\frac{\partial L}{\partial i} = \frac{\partial L}{\partial R} < 0$ and

$\frac{\partial M}{\partial i} = \frac{\partial M}{\partial R} > 0$. In the second phase (i.e. for $i \in (P, Q)$), the properties of the money demand and the money supply are precisely the reversal, i.e. $\frac{\partial L}{\partial R} > 0$ and $\frac{\partial M}{\partial R} < 0$. The second phase can be interpreted as describing economic subjects behaviour in a situation resembling a liquidity trap. The relaxation oscillations in this case represent the relaxation oscillations on the money market, i.e. on the LM side of the IS-LM model. The seemingly unexpected fluctuations of the long-term real interest rate R are caused by the relaxation oscillations on the money market. Furthermore, in [O4], we also model an economic situation where the courses of the functions $L(i)$ and $M(i)$ are divided into more than three parts, and where the phase characterised by an usual behaviour of the money demand and of the money supply alternates with the phase characterised by precisely the reverse behaviour of the money demand and of the money supply (this latter behaviour can be interpreted as describing economic subjects in a situation resembling a liquidity trap). In this case, several seemingly unexpected fluctuations of the long-term real interest rate can emerge, see e.g. the vertical segments A, B, C, D, E and F in Figure 6. The relaxation oscilla-

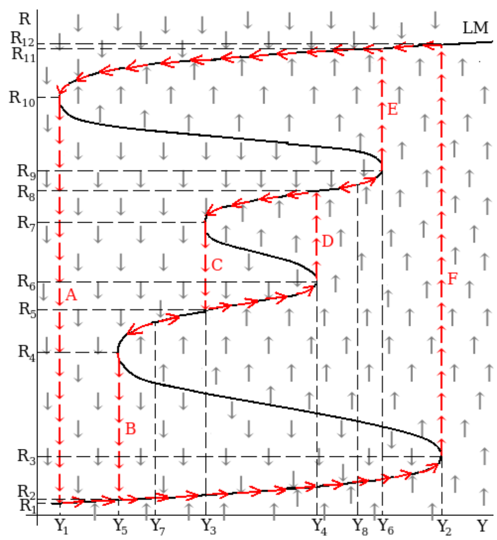

FIGURE 6. Relaxation oscillations on the money market where the courses of the functions $L(i)$ and $M(i)$ are divided into seven parts, three of them being interpreted as a liquidity trap (see [O4])

tions on the money market (i.e. these fluctuations of long-term real interest rate) emerge as a consequence of the fiscal policy. However, this phenomenon is not desirable. In order to reduce the impact of the latter, we have suggested a possible cooperation between the fiscal and monetary policy, see [O4].

In the paper [P3], we show that Devaney, Li-Yorke and distributional chaos in economies, described by the IS-LM/QY-ML model, may arise. We consider the IS-LM/QY-ML model where the functions I, S, L, M, Q, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{T}$ satisfy conditions (10), (11), (13), (14), (15) and, in addition, the conditions $\frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y}$ and $\frac{\partial Q}{\partial Y} < 1$. Under these conditions, the corresponding IS and QY curves are in fact graphs of decreasing functions, and the corresponding LM curve (which is identical to the ML curve) is in fact a graph of an increasing function. These properties of the IS, QY and LM (ML) curves are in a sense generic. It can be proved that the singular point (if it exists) of the modified IS-LM model with previously mentioned conditions on the functions I, S, L and M is a stable node or a stable focus, and, similarly, the singular point (if it exists) of the QY-ML model with previously mentioned conditions on the functions Q, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{T}$, M and L is an unstable saddle point, see [P3]. The phase portrait of the IS-LM/QY-ML model corresponding to this situation is displayed in Figure 7. In Figure 7 the black curves and trajectories correspond to the IS-LM model, i.e.

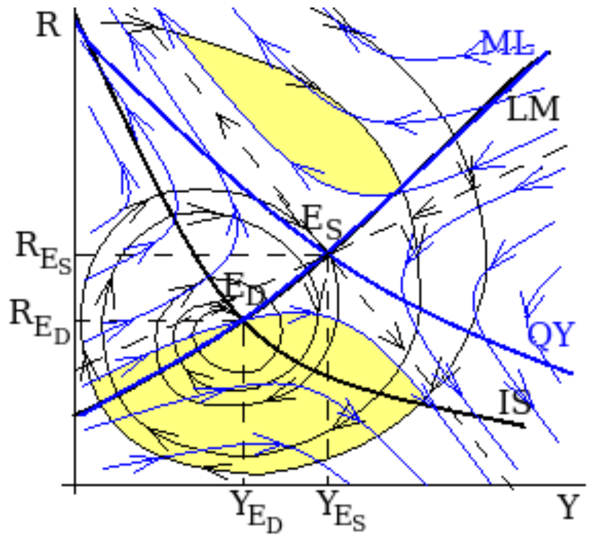

FIGURE 7. An illustration of the chaotic sets in the IS-LM/QY-ML model (see [P3])

to the IS-LM branch of the IS-LM/QY-ML model, where an stable focus arises, and the blue curves and trajectories correspond to the QY-ML model, i.e. to the QY-ML branch of the IS-LM/QY-ML model, where unstable saddle emerges. Yellow areas in Figure 7 represent the chaotic sets whose existence for the IS-LM/QY-ML model is admitted by Theorems 2.1 and 2.2. Furthermore, if the members of the set of solutions V^* (corresponding to a chaotic set V) to the IS-LM/QY-ML model are interpreted as describing the course of an economy influenced by a regular economic cycle or describing the course of an economy influenced by such an economic cycle in which two consecutive recession phase durations and two consecutive expansion phase durations are of different lengths, then there exist Devaney, Li-Yorke and distributional chaos in this economy, see the proof of Theorem 2.3 and its economic interpretation in [P3].

5. CONCLUSION

The IS-LM model is one of the cornerstones of the modern macroeconomic modelling. Since the formulation of the original IS-LM model by Hicks in 1937 (see [10]) many papers (see e.g. [7], [8], [13], [20], [23], [22], [26] or [27]) have been focused on the study of the various types of behaviour of modifications of this model (three-sector models, bifurcations, limit cycles etc.). In the present thesis, we discovered a new possibility of a dynamical behaviour in the original, the modified and the extended two-dimensional model. While the original model describes the economies only in the stage of a recessionary gap, our modifications extend the applicability of this model also to the remaining stages, in particular, the last presented model describes the macroeconomic situation during the whole economic cycle. Looking for these models and the study thereof led us to examine and extend a new approach, based on the Euler equation branching, to the description of the dynamical behaviour in the plane $\mathbb{R}^2$. This enabled us to prove certain results on the possible inception of chaos in $\mathbb{R}^2$.

Presentations related to the thesis

[C1] 37th International Conference Macromodels 2010, Pułtusk, Poland, December 1 - 4, 2010.
Talk: Mathematical Modelling of Macroeconomic Equilibrium or IS-LM Model Based on Special Functions.

[C2] 29th International Conference on Mathematical Methods in Economics 2011, Jánská Dolina, Slovakia, September 6 - 9, 2011.
Talk: Dynamical Behaviour and Existence of Chaos in the Modifications of Macroeconomic IS-LM Model.

[C3] 38th International Conference Macromodels 2011, Poznań, Poland, November 30 - December 3, 2011.
Talk: Potential Existence of Devaney. Li-Yorke and Distributional Chaos in two Modifications of Macroeconomic IS-LM Model.

[C4] Tenth Workshop on Interactions between Dynamical Systems and Partial Differential Equations 2012 (JISD2012), Barcelona, Spain, May 28 - June 1, 2012.
Talk: Relaxation Oscillations Emerging in Macroeconomics.

[C5] 30th International Conference on Mathematical Methods in Economics 2012, Karviná, Czech Republic, September 11 - 13, 2012.
Talk: Models of Seemingly Unexpected Behaviour of Aggregate Income or Real Interest Rate.

[C6] 39th International Conference Macromodels 2012, Zakopane, Poland, December 5 - 8, 2012.
Talk: Relaxation Oscillations Emerging on Money or Financial Assets Market or Model of Unexpected Fluctuations of Long-Term Real Interest Rate.

[C7] International Conference Equadiff 13 (2013), Praha, Czech Republic, August 26 - 30, 2013.
Talk: Chaotic behaviour of continuous dynamical system generated by Euler equation branching in plane $\mathbb{R}^2$ and its application in macroeconomics.

[C8] Symposium on Differential and Difference Equations 2014, Homburg/Saar, Germany, September 5 - 8, 2014.
Talk: Existence of Chaos in the Plane $\mathbb{R}^2$ and its Application in Macroeconomics.

Publications constituting the body of the thesis

[P1] B. Volná Kaličinská, *Augmented IS-LM model based on particular functions*, Appl. Math. Comput. 219 (3) (2012), 1244–1262.

[P2] B. Volná, *A dynamic IS-LM model with relaxation oscillations*, Applicable Analysis (2015), DOI:10.1080/00036811.2015.1026336.

[P3] B. Volná, *Existence of chaos in the plane $\mathbb{R}^2$ and its application in macroeconomics*, Appl. Math. Comput. 258 (2015), 237–266.

Other publications

[O1] Barbora Kaličinská, *Mathematical modelling of macroeconomic equilibrium or IS-LM model based on special functions*, Macromodels 2010 - proceedings of the Thirty Seventh International Conference (2011), 21–32.

[O2] Barbora Volná Kaličinská, *Potential existence of Devaney, Li-Yorke and Distributional chaos in two modifications of macroeconomic IS-LM model*, Macromodels 2011 - proceedings of the Thirty Eighth International Conference (2012), 15–24.

[O3] Barbora Volná (Kaličinská), *Models of unexpected fluctuations of aggregate income or real interest rate*, proceedings of the 30th International Conference Mathematical Methods in Economics (2012), 974–979.

[O4] Barbora Volná, *Can we explain unexpected fluctuation of long-term real interest rate?*, arXiv preprint: arXiv:1211.2709. Submitted.

[O5] Barbora Volná, *Chaotic Behaviour of Continuous Dynamical System Generated by Euler Equation Branching and its Application in Macroeconomic Equilibrium Model*, Mathematica Bohemica (Equadiff 13). Accepted.

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