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Abstract of the Ph.D. Thesis

Various types of chaos and entropy in dynamical systems

Author:
Mgr. Zuzana Roth

Supervisor:
doc. RNDr. Zdeněk Kočan, Ph.D.

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Autor: Mgr. Zuzana Roth

Školitel: doc. RNDr. Zdeněk Kočan, Ph.D.

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Dizertant: Mgr. Zuzana Roth
Matematický ústav v Opavě, Slezská univerzita v Opavě

Školitel: doc. RNDr. Zdeněk Kočan, Ph.D.
Matematický ústav v Opavě, Slezská univerzita v Opavě

Školící pracoviště: Matematický ústav v Opavě, Slezská univerzita v Opavě

Oponenti: doc. RNDr. Marta Štefánková, Ph.D.
Matematický ústav v Opavě, Slezská univerzita v Opavě

doc. RNDr. Roman Hric, PhD.
Fakulta prírodných vied, Univerzita Mateja Bela, Banská Bystrica

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Předseda oborové rady: prof. RNDr. Miroslav Engliš, DrSc.
Matematický ústav v Opavě
Slezská univerzita v Opavě
Na Rybníčku 626/1
74601 Opava

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1 Introduction

Over the history of mankind we have always tried to determine what can be considered as order and what is already chaos. And so nowadays we have dozens of different definitions and levels of chaotic behaviour. It is not in the scope of this thesis to even name all of them (we will closely look at 3 different areas).

The history of chaos theory in mathematics could go as far back as Lorenz (butterfly effect, 1960s), Poincaré (sensitivity to initial conditions, n-body problem, 19th century), or even Kepler (planetary motion, 17th century)... but as Oestreicher said in [Oes07] *"Lorenz had rediscovered the chaotic behavior of a nonlinear system, that of the weather, but the term chaos theory was only later given to the phenomenon by the mathematician James A. Yorke, in 1975."* It is true that Yorke, together with Li, gave one of the first definitions of chaos in mathematics in [LY75], even though more commonly we hear Devaney's description used as the definition of chaos [Dev89].
The term entropy has been used in physics to describe chaotic behavior since approximately the 19th century (see [Cla56, Cla67]), but the popularity of this term came with information science in the mid 20th century [SWB51] while in math the attention was brought to entropy by Kolmogorov and Sinai in [Kol58, Sin59]. The term entropy can now be found in many areas of science; usually it means disorder or chaos. In mathematics a system with positive entropy can also be called h-chaotic. We can try to dig deeper in history for other hints of chaos definitions, but if we look at it closely, chaos theory and dynamical systems are relatively young branches of mathematics with good potential. For some more details about the history of different types of chaos and entropy see [Dow07, Oes07, CAM$^+$05] and [Wal00, chapters: 4, 7, 8].

Out of the many different definitions of chaos and entropy, this thesis will mainly consider distributional chaos as defined in [SS94, BSŠ05] (and also its relation with Li-Yorke chaos [LY75]). For entropy we will focus on topological entropy [AKM65] and a special case of measure-theoretic entropy - "fair entropy" originally defined in [MR18].

The thesis is based on 4 papers [A, B, C, D]. The first 2 are focused more on distributional chaos, while the other 2 deal with entropy. The main topic of this thesis involves the problem of determining the relationships between the different types of chaos, as well as the "stability" of each type. Stability can be understood as the persistence of chaos under: conjugacy, composition, extension, small pertubations, etc. The topic of stability also raises the question of topological invariants, which are not always easy to find (2 of my 4 articles introduce 2 new invariants [A, C]).

The thesis is organized as follows. In the next section we give motivational comments and relations about the main types of chaos for this thesis. In section 3 we properly introduce the necessary notations, background and definitions. Section 4 points out our main results from all 4 articles. Open questions follow in section 5. Then we close with a list of publications, citations and presentations. In the appendices the reader can find full copies of the articles [A, B, C, D] along with statements confirming my coauthorship and the acceptance of article [C] for publication.

2 Motivation for the research

If we look at continuous functions acting on the closed interval, a lot of the definitions of chaos coincide. Li and Yorke claimed that period 3 implies chaos [LY75], while Smítal (and Misiurewicz) showed that period 3 is actually not necessary for Li-Yorke (LY) chaos [MS88, Smí86] already on the interval (even for C^∞ maps). However their examples also have topological entropy equal to 0. That led to an assumption that positive topological entropy is a stronger type of chaos than LY. When later Schweizer and Smítal defined distributional chaos [SS94] (abbreviated DC, or DC1, since later there were defined DC1, DC2, DC3 as subtypes of DC - for details see section 3), they also discoverd that the existence of a single DC-pair is equivalent to positive topological entropy for continuous maps on a closed interval. Later it also was shown that the existence of any kind of DC-pair is equivalent to the existence of an uncountabele DC1-set and positive topological entropy for continuous maps on

closed intervals and graphs, even though in general these concepts are distinguishable (see eg. [BSŠ05, Pik07, Mál07, KKKM11, Koč12]). It is also known that topological entropy is a topological invariant, and the same was shown about DC1 and DC2 (see [Wal00, SŠ04]).
And so the question naturally arose how much "weaker" is the definition of DC3 in general spaces than the first 2 types? Well, it turned out that even though on the closed interval DC3 is equivalent to positive topological entropy, in general it is not even as strong as LY, moreover the property, as it is defined, is in some sense unstable [A]. Nevertheless, we had hoped to carry at least some of the results from intervals and graphs to dendrites, since they are often understood as a collection of intervals glued together, and so they are a natural next step from the interval and graphs to slightly more complicated spaces. The results of that investigation are found in [B].

On the other hand, a common task when studying entropy is to calculate its value, or at least find some upper or lower bounds. (Lower bounds are best when we want to show that the entropy is positive, and upper when we want to show that it is zero.) How can we connect these entropy bounds with the dimension, the metric, or other properties of the system? In the process of dealing with these questions, we were able to improve some older results from [DZG98], and as a bonus we found a new topological invariant [C].

Fair measures and fair entropy are the newest concepts we worked with. Both are introduced in [MR18] in an attempt to find (again) an easier way to calculate topological entropy. But instead of producing a new formula for topological entropy they lead instead to a lower bound ($h_{\text{fair}} \leq h_{\text{top}}$) and a new topological invariant. We looked at the challenge of generalizing this concept; we managed to go beyond compact spaces and continuous mappings [D]. Moreover, just as Birkhoff's ergodic theorem describes forward trajectories, fair measures give us a tool for studying backward trajectories.

3 Terminology and notation

If not indicated otherwise, we will use the following notation throughout the thesis. The pair (X, d) will be a non-empty compact metric (or at least metrizable) space with metric d. By $f : X \to X$ we denote a continuous map, and f^n denotes the nth iterate of f, for $n \in \mathbb{N}_0$, so that $f^0(x) = id(x) = x$, $f^{n+1} = f \circ f^n$.

Dendrites are locally connected continua (nonempty, compact, connected metric spaces) not containing any simple closed curves.

The concept of a *tame graph (dendrite)* was introduced in [BBP+18]. Let $E(G)$ denote the *endpoints* of the continuum G (i.e. the points $x \in G$ having arbitrarily small neighborhoods V with one-point boundaries $\#\partial V = 1$). Let $B(G)$ denote the *branching points* (i.e. the points $x \in G$ such that any sufficiently small neighborhood V of x has at least three points in its boundary $\#\partial V \geq 3$). G is called a *tame graph* if $E(G) \cup B(G)$ has countable closure.

As an *expansive non-invertible map* we consider Walter's definition from [Wal00]: A continuous map $f : X \to X$ of a compact metric space (X, d) is called *positively expansive* if there is $c > 0$ (an *expansivity constant*) such that if $d(f^n(x), f^n(y)) \leq c$ for all $n \geq 0$, then $x = y$.

A metric $d : X \times X \to [0, \infty)$ is *compatible* if the topology it induces coincides with the topology of X (where X is a compact metrizable space).

If $f : X \to X$ is a continuous map, then $\mathcal{M}(X, f)$ denotes the space of all f-invariant probability measures on X. If $\mathcal{A}, \mathcal{B}$ are two partitions of X, then their common refinement is $\mathcal{A} \bigvee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Li-Yorke chaos

As was already mentioned, one of the oldest definitions of chaos is attributed to Li and Yorke [LY75] and so we can not omit this definition either. A pair of distinct points $(x, y) \in X^2$ is called *Li-Yorke* (LY) if

$$\liminf_{k\to\infty} d(f^k(x), f^k(y)) = 0 \quad \textit{and} \quad \limsup_{k\to\infty} d(f^k(x), f^k(y)) > 0. \tag{1}$$

A subset $S \subset X$ is *LY-scrambled* if it contains at least 2 distinct points and every pair of distinct points in S is LY. Originally, to call a system LY-chaotic, there needed to be an uncountable LY-scrambled set (LY_u), later, especially in more general spaces, there arose a discussion about LY-pairs (LY_2) or infinite (but not uncountable) LY-scrambled sets (LY_∞). This distinction is not necessary on the closed interval or graphs, since the existence of a LY-pair implies the existence of an uncountable LY-set, but this is not true in general [RS14, KS89, FPS95].

Distributional chaos

From the definition of LY-chaos we can see that even though for a system to be called LY-chaotic we need that the trajectories of 2 points will be sometimes close and sometimes farther apart, one or the other can happen very "rarely" in time. As an alternative, another type of chaos was defined in [SS94] where we look not just if the separation and approach of trajectories happen, but also the proportion of times when they happen. This type is called nowadays distributional chaos of type 1 (or DC1 for short). Later this chaos was divided into 3 different types, DC1, DC2 and DC3 (see [BSŠ05]) which are different in general, but the same in the interval. In [A] we showed that DC3 can be a really weak and unstable type of chaos, so we proposed a better definition and called it DC2$\frac{1}{2}$, which, as was shown, fixed the problems of DC3, but in general DC2$\frac{1}{2}$ is essentially weaker than DC2. (There is another kind of distributional chaos, DC1$\frac{1}{2}$, defined in [Dow14], but this thesis will not discuss it.)

Distribution functions: For a pair (x, y) of points in X we define the *upper distribution function* ($\Phi^*_{(f,x,y)}(\delta)$) and the *lower distribution function* ($\Phi_{(f,x,y)}(\delta)$) generated by f as

$$\Phi^*_{(f,x,y)}(\delta) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \le k \le n; d(f^k(x), f^k(y)) < \delta\}, \tag{2}$$

and

$$\Phi_{(f,x,y)}(\delta) = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \le k \le n; d(f^k(x), f^k(y)) < \delta\}, \tag{3}$$

where $\#A$ denotes the cardinality of the set A. If it is clear from context that the distribution functions are generated by f, it is often omitted in the notation.

DC1: A pair $(x, y) \in X^2$ is called *distributionally scrambled of type 1* if

$$\Phi^*_{(f,x,y)}(\delta) = 1, \text{ for every } 0 < \delta \leq \operatorname{diam} X \quad \textit{and}$$

$$\Phi_{(f,x,y)}(\varepsilon) = 0, \text{ for some } 0 < \varepsilon \leq \operatorname{diam} X.$$

DC2: A pair $(x, y) \in X^2$ is called *distributionally scrambled of type 2* if

$$\Phi^*_{(f,x,y)}(\delta) = 1, \text{ for every } 0 < \delta \leq \operatorname{diam} X \quad \textit{and}$$

$$\Phi_{(f,x,y)}(\varepsilon) < 1, \text{ for some } 0 < \varepsilon \leq \operatorname{diam} X.$$

DC3: A pair $(x, y) \in X^2$ is called *distributionally scrambled of type 3* if

$$\Phi_{(f,x,y)}(\delta) < \Phi^*_{(f,x,y)}(\delta),$$

for every δ in some interval (a, b), where $0 \leq a < b \leq \operatorname{diam} X$.

DC$2\frac{1}{2}$: We can define distribution functions at 0 as limits:

$$\Phi_{(f,x,y)}(0) = \lim_{\delta \to 0^+} \Phi_{(f,x,y)}(\delta) \quad \text{and} \quad \Phi^*_{(f,x,y)}(0) = \lim_{\delta \to 0^+} \Phi^*_{(f,x,y)}(\delta).$$

Then $(x, y) \in X^2$ is called *distributionally scrambled of type* $2\frac{1}{2}$ if

$$\Phi_{(f,x,y)}(0) < \Phi^*_{(f,x,y)}(0).$$

A subset S of X is *distributionally scrambled of type i* (or a DCi set), where $i \in \{1, 2, 2\frac{1}{2}, 3\}$, if every pair of distinct points in S is a DCi pair. Originally, the dynamical system (X, f) was called *distributionally chaotic of type i* (a DCi system), if there was a DCi pair (DCi_2), later the focus was moved to uncountable DCi-sets (DCi_u).

Entropy and related topics

We mostly do not work directly with the definition of entropy, but as a general reminder we recall at least parts of the definitions. For a more detailed picture of measure theoretic entropy see [Wal00, chapter 4] and for topological entropy see [Wal00, chapters: 7, 8].

Topological entropy: The topological entropy of a map f is commonly denoted as $h(f)$ or if there can be confusion $h_{\mathrm{top}}(f)$.

The topological entropy of the map f is defined by the formula (Bowen-Dinaburg definition)

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon) \right),$$

where $s_n(\varepsilon)$ denotes the maximal cardinality of an $(n, \varepsilon) -$ *separated* set. A subset K of X is said to be $(n, \varepsilon) -$ *separated* if each pair of distinct points of K is at least ε apart in the metric d_n, where the metric d_n is defined for each positive integer n by the formula: $d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i < n\}$.

Measure theoretic entropy: The measure theoretic entropy of a map f is usually denoted as $h_\mu(f)$, where μ is the corresponding invariant measure.

The measure theoretic entropy of the map f for an invariant measure μ is defined by the formula

$$h_\mu(f) = \sup_{\mathcal{X}} \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i} \mathcal{X} \right), \quad \text{where } H(\mathcal{A}) = -\sum_{A \in \mathcal{A}} \mu(A) \cdot \log \mu(A),$$

where the supremum is taken over all countable measurable partitions $\mathcal{X}$ of X such that $H(\mathcal{X})$ is finite. $H(\mathcal{X})$ is called the static entropy of the partition.

If f is a continuous map of a compact metric space X, then by the variational principle [Wal00]

$$h_{\mathrm{top}}(f) = \sup \{h_\mu(f) : \mu \in \mathcal{M}(X, f)\}.$$

Fair entropy: Fair entropy is a measure theoretic entropy based on a special (fair) measure, it was originally defined in [MR18], as the entropy of the unique fair measure, but we extended the concept in [D] and so generalized the definition too. We denote the fair entropy as $h_{\mathrm{fair}}(f)$.

In our case, the space X does not need to be compact, and f does not need to be continuous. We assume that X is a Polish space (separable completely metrizable topological space) and $\mathcal{F}$ is the Borel σ-algebra. We also assume that $f : X \to X$ is a surjection and $\{X_i\}_{i=1}^{\infty}$ is a partition of X such that each restriction $f|_{X_i}$ is a measurable isomorphism onto its image.

An invariant measure $\mu \in \mathcal{M}(X, f)$ is called *fair* if each measurable set $B \prec \mathcal{A}$ (B is a subset of an element of $\mathcal{A}$) satisfies

$$\mu(X_i \cap f^{-1}(B)) = \frac{\mu(B)}{c(B)}, \text{ for all } i \in p(B). \tag{4}$$

Here $\mathcal{A} = \bigvee_{i=1}^{\infty} \{f(X_i), X \setminus f(X_i)\}$ is a countable measurable partition, and for a set $B \prec \mathcal{A}$ we define $p(B) := \{i; B \subset f(X_i)\}$ and $c(B) := \#p(B)$. The number $c(B)$ (or $c(x)$ for the singleton $\{x\}$) is always positive since f is surjective, but may be infinite.

The *fair entropy* of a system (X, f) is the supremum of measure-theoretic entropies of its fair measures. (If the system has no fair measures, we take the supremum over the empty set to be zero.)

$$h_{\text{fair}}(f) = \sup \{h_\mu(f) \mid \mu \in \mathcal{M}(X, f) \text{ is fair}\}.$$

HausLip constant: To research the connection of topological entropy and other properties of a system (X, f) (X is a compact metrizable space and f is continous) in [C], we define the *HausLip constant* of the system (X, f) as follows:

$$\text{HausLip}(X, f) := \inf_{d \in \mathcal{D}(X)} \text{HD}_d(X) \cdot \log^+ \text{Lip}_d(f), \tag{5}$$

where $\mathcal{D}(X)$ is the set of all metrics on X compatible with its topology; $\log^+(y) = \max\{\log(y), 0\}$. Furthermore we make the agreement that $\infty \cdot 0 = 0$ and $0 \cdot \infty$ is undefined in (5). Also, $\inf(\emptyset) = \infty$, so that (5) is defined for every system.
$\text{HD}_d(X)$ denotes the *Hausdorff dimension* of X with respect to the metric d,

$$\text{HD}_d(X) = \inf\{s \geq 0 \mid \mu^s(X) = 0\},$$

where the limit $\mu^s(X) = \lim_{\varepsilon \to 0} \mu^s_\varepsilon(X)$, called *$s$-dimensional Hausdorff measure* is well-defined as ε decreases, $\mu^s_\varepsilon(X) = \inf \sum_i |B_i|^s_d$ where the infimum is taken over all possible ε-covers; $|B|_d = \sup\{d(x, y) \mid x, y \in B\}$ denotes the diameter of a subset $B \subset X$ under the metric d and an *ε-cover* of X is a collection of sets B_i each of diameter $\leq \varepsilon$ whose union equals X. (For more details about Hausdorff dimension see [BBT97, chapter 3.8] or [Mor19, Appendix 4].)

$\mathrm{Lip}_d(f)$ denotes the *Lipschitz constant* of f with respect to the metric d,

$$\mathrm{Lip}_d(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

4 Main Results

[A] On the weakest version of distributional chaos.

As we mentioned before, the interest in this article was the investigation of the "strengths and weaknesses" of DC3.

Theorem 1 ([A], Th. 1)**.** There exists a distal dynamical system which is DC3 chaotic. Thus, DC3 chaos does not imply Li-Yorke chaos.

Theorem 2 ([A], Th. 2-3)**.** Neither the existence of DC3-pairs nor the existence of an uncountable DC3-set is preserved by topological conjugacy.

Since DC3 was such a weak form of chaos we introduced DC$2\frac{1}{2}$ chaos with better properties.

Theorem 3 ([A], Sec. 5)**.** Let f and g be topologically conjugate continuous maps of a compact metric space (X, d). Then f is DC$2\frac{1}{2}$ if and only if g is DC$2\frac{1}{2}$. Moreover if $(x_1, x_2) \in X^2$ is DC$2\frac{1}{2}$, then it is LY.

[B] Distributional chaos and dendrites.

Since all the known examples for showing that the different types of DC are distinct were in more complicated spaces, and the definitions coincide on intervals and graphs (as we mentioned in section 2), we had a hope that dendrites would follow this pattern. Our discovery was that dendrites are the simplest known spaces for distinguishing all of the basic types of chaos.

Theorem 4 ([B], Sec. 3)**.** DC$(i+1)$ does not imply DC(i) on dendrites in the sense of uncountable sets or pairs ($i \in \{1,2\}$).

Theorem 5 ([B], Sec. 3)**.** DC3 does not imply DC$2\frac{1}{2}$ on dendrites in the sense of pairs.

Note. For proving that the existence of an uncountable DC3-set does not imply existence of DC2-pairs, we used a dendrite built on a subshift of the full 5-shift (see Figure 1) and the following lemma:

Lemma 1 ([B], Sec. 3.5)**.** If $Y \subset \{0,1,2,3,4\}^{\mathbb{N}_0}$ is a subshift then there is a subdendrite $\mathcal{G}_{5_Y}$ of the dendrite $\mathcal{G}_5$ invariant under g_5. Let $\mathcal{E}_{5_Y}$ be the set of end points of $\mathcal{G}_{5_Y}$, then $(\mathcal{E}_{5_Y}, g|_{\mathcal{E}_{5_Y}})$ is topologically conjugate to (Y,σ) and all DC-pairs of $\mathcal{G}_{5_Y}$ are contained in $\mathcal{E}_{5_Y}$.

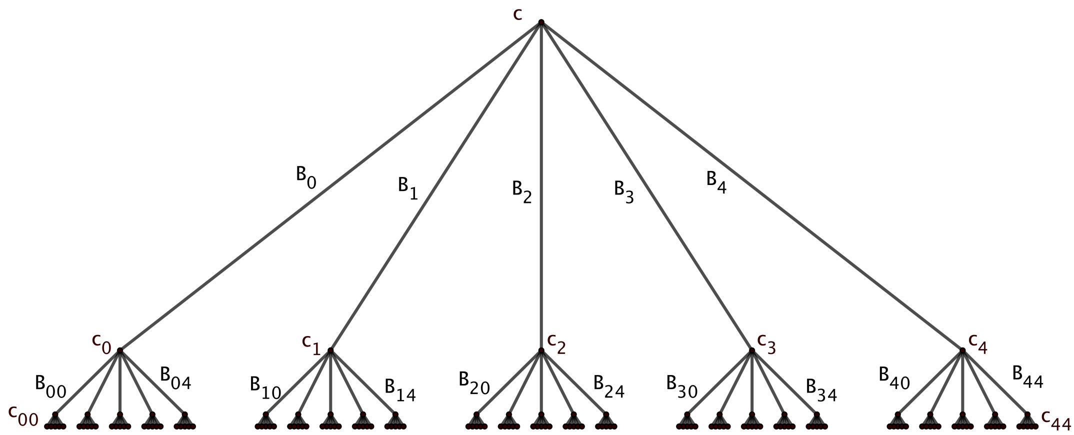

Figure 1: Gehman style dendrite $\mathcal{G}_5$

In fact this shows us a little bit more: we can build a Gehman-style dendrite and subdendrite "on" any shift (or subshift) with a finite alphabet and the dynamics will be basically the same as we are used to on shift spaces. And so dendrites are much more complicated spaces than our intuition might suggest.

Theorem 6 ([B], Sec. 3)**.** Existence of a DCi pair does not imply existence of an infinite DCi-set for any known $i-$type of distributional chaos $\left(i \in \left\{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3\right\}\right)$.

In the end of the article we were able to get some positive results and show that the existence of an arc horseshoe implies the strongest type of DC on dendrites.

Theorem 7 ([B], Sec. 4)**.** Let f be a continuous self-map of a dendrite. If an iterate of f has an arc horseshoe then f is $\mathrm{DC1}_u$ and ω-chaotic.

[C] Inequalities for entropy, Hausdorff dimension, and Lipschitz constants.

In this article we construct suitable metrics for two classes of topological dynamical systems in order to get a lower bound for topological entropy in terms of the resulting Hausdorff dimensions and Lipschitz constants. As a nice side product we define also a new topological invariant. (Recall that the HausLip constant is defined in (5).)

Lemma 2 ([C], Lem. 2)**.** The HausLip constant is an invariant of topological conjugacy and is bounded below by the topological entropy.

Theorem 8 ([C], Th. 3)**.** Let f be a linear map of the n-torus $\mathbb{R}^n/\mathbb{Z}^n$. Then the HausLip constant equals the entropy. In other words, for each $\varepsilon > 0$ there is a metric d on $\mathbb{R}^n/\mathbb{Z}^n$ compatible with the topology such that

$$\mathrm{HD}_d(\mathbb{R}^n/\mathbb{Z}^n) \cdot \log^+ \mathrm{Lip}_d(f) < h_{\mathrm{top}}(f) + \varepsilon.$$

Theorem 9 ([C], Th. 5)**.** If $f : X \to X$ is positively expansive, then $\mathrm{HausLip}(X, f) = h_{\mathrm{top}}(f)$. In other words, for every $\varepsilon > 0$ there is a metric d on X compatible with its topology such that

$$\mathrm{HD}_d(X) \cdot \log^+ \mathrm{Lip}_d(f) < h_{\mathrm{top}}(f) + \varepsilon.$$

As a corollary, we get also a one-sided version of Mañé's theorem for expansive homeomorphisms [Mañ79].

Corollary 10 ([C], Cor. 6)**.** Any space X admitting a positively expansive map f has finite topological dimension. If f can be chosen with zero entropy, then X is totally disconnected.

[D] Fair measures for countable-to-one maps.

We recall that the results in this section go beyond continuous maps and compact spaces; for the setting and notation see section 3: fair entropy.
Since we generalized the notion of fair measures and entropy, we also generalized the results from [MR18].

Theorem 11 ([D], Sec. 3)**.** If X is a Polish space, f is a piecewise continuous map on X, $\{X_i\}_i$ is countable, and μ is an ergodic fair measure, then

(a) For every $\phi \in L^1(X)$ for μ-almost every $x_0 \in X$ for almost every backward trajectory (x_n) of x_0,

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(x_n) \to \int_X \phi \, d\mu.$$

(b) If also X is compact, then for μ-almost every $x_0 \in X$, almost every backward trajectory (x_n) of x_0 equidistributes for μ.

(c) If also f has a one-sided generating partition of finite entropy, then for μ-almost every $x_0 \in X$, for almost every backward trajectory (x_n) of x_0,

$$\frac{1}{N}\sum_{n=0}^{N-1} \log c(x_n) \to \int \log c(x)\, d\mu(x) = h_\mu(f).$$

For finding fair measures and fair entropy on countable state Markov shifts, we used knowledge from probability theory and shift spaces. We formed a stochastic matrix Q with entries $q_{ji} = \frac{m_{ij}}{c_j}$, where $c_j = \sum_i m_{ij}$ and m_{ij} are the entries from the transition matrix for the Markov shift.

Theorem 12 ([D], Sec. 4)**.** Let (Σ_M, σ) be a transitive countable-state Markov shift with all c_j finite. Given any point $y_0 \in \Sigma_M$ the behavior of a random backward trajectory (y_n) is as follows:

(a) If Q is positive recurrent, then there is a unique fair measure μ and (y_n) equidistributes for μ.

(b) If Q is null recurrent, then (y_n) is dense in Σ_M, but visits each cylinder set $[i]$ with limiting frequency zero. There is no fair measure.

(c) If Q is transient, then (y_n) visits each cylinder set $[i]$ only finitely often. There is no fair measure.

To get further results we needed a good notion of isomorphism and the classic concept was not appropriate for fair measures. We proposed the following definition:

Two systems (X_1, f_1), (X_2, f_2) are called *fair-isomorphic* if there exist totally invariant countable sets $N_1 \subset X_1$, $N_2 \subset X_2$ and a bijection (called the *fair isomorphism*) $\psi : X_1 \setminus N_1 \to X_2 \setminus N_2$, bimeasurable with respect to the Borel σ-algebras, such that $f_2 \circ \psi = \psi \circ f_1$.

Theorem 13 ([D], Th. 5.2)**.** A fair isomorphism induces an entropy-preserving bijection of non-atomic fair measures. Moreover, this implies that fair-isomorphic systems have the same fair entropy.

Equipped with the right isomorphism, we were able to connect interval maps and tame graph maps back to Markov shifts (for details see [D, section 6-8]). The connection is given by the itinerary map $i(x) = (i_0\ i_1\ i_2 \cdots) \in \mathcal{X}^{\mathbb{N}_0}$ where $f^n(x) \in i_n \in \mathcal{X}$ for all n.

Theorem 14 ([D], Sec. 6-7)**.** Let f be a mixing interval map with a countable Markov partition $\mathcal{X}$. The itinerary map $i : [0,1] \to \mathcal{X}^{\mathbb{N}_0}$ gives a fair isomorphism of f with the associated Markov shift.

This gives us all nonatomic fair measures for f. This also gives us the behaviour of typical backward trajectories. In the positive-recurrent case, we also give an algorithm to construct a conjugate interval map for which Lebesgue measure is fair.

We got similar results for tame graphs.

Theorem 15 ([D], Sec. 8)**.** Let g be a mixing tame graph map with a countable Markov partition $\mathcal{X}$. Then the itinerary map is a fair isomorphism. Moreover there is a piecewise continuous interval map f which is fair isomorphic to g.

We call this map f a *cut-and-paste model* for g.

5 Open problems

Some interesting ideas for further research are given below, organised by topic. Some of the questions listed here can be found in the articles, but some of them are new.

Distributional chaos

Question 1. Does DC3 imply Li-Yorke chaos on dendrites?

The question can be understood in two senses. Does an uncountable DC3-set imply existence of an uncountable LY-set? Does a DC3-pair imply existence of a LY-pair?

We already know that these implications do not hold in general. But if we ask about the interval and graphs, they do hold, and all of the known examples of dendrite maps with DC3 also have LY. My guess is that it will be possible to construct a dendrite map with an uncountable DC3-set and no uncountable LY-set, but the dendrite will need to be non-tame (in the sense of [BBP$^+$18]). An example with a DC3-pair but no infinite LY-set is already given in [Drw18].

If the answer to this question turns out to be negative, it will give us one of the simplest systems for which this implication does not hold.

Subquestion 1.1. Does DC3 imply type $2\frac{1}{2}$ distributional chaos (DC$2\frac{1}{2}$) on dendrites for uncountable sets?

A negative answer to the preceding question would automatically solve this question as well, since we know that DC$2\frac{1}{2}$ implies LY [A]. But if the answer to the preceding question turns out to be positive, it could be that the definition of DC$2\frac{1}{2}$ is not actually needed on dendrites. Nevertheless my guess is that the answer is again negative, since as we showed in [B], the implication does not hold for pairs.

Question 2. Does DC$2\frac{1}{2}$ imply DC2 on dendrites?

This question is important to determine if the definition of DC$2\frac{1}{2}$ is needed on dendrites, or if it is covered by one of the other types of DC.

HausLip and topological entropy

Since the origin of the HausLip constant was connected to topological entropy, we would like to see several properties which are known about topological entropy to be true about the HausLip constant as well.

Question 3. Does the HausLip constant "inherit" properties of topological entropy, namely: do the product rule, iteration rule or factor rule for topological entropy hold for the HausLip constant? To be precise, is it true that

- $\mathrm{HausLip}(X \times Y, f \times g) = \mathrm{HausLip}(X, f) + \mathrm{HausLip}(Y, g)$,
- $\mathrm{HausLip}(X, f^n) = n \cdot \mathrm{HausLip}(X, f)$,
- $\mathrm{HausLip}(X, f) \leq \mathrm{HausLip}(Y, g)$ when (X, f) is a factor of (Y, g)?

In our paper [C] we showed several examples and conditions, where $\mathrm{HausLip}(X, f) = h_{\mathrm{top}}(f)$ (and one example on the Hilbert cube where $\mathrm{HausLip}(X, f) > h_{\mathrm{top}}(f)$). But there is also the opposite question, which remains unanswered so far:

Question 4. Is there an interval map $f : I \to I$ with $\mathrm{HausLip}(I, f) > h(f)$?

Fair measures and fair entropy

Since the topic is new, there are many open questions in this field. I indicate just a few of the more basic questions:

Question 5. Can we carry the properties of topological entropy over to fair entropy:

- Is it true that $h_{\mathrm{fair}}(f^n) = n h_{\mathrm{fair}}(f)$, or how does $h_{\mathrm{fair}}(f^n)$ relate to $h_{\mathrm{fair}}(f)$?
- Is it true that $h_{\mathrm{fair}}(f \circ g) = h_{\mathrm{fair}}(g \circ f)$ as is known about h_{top}?

Question 6. Does fair entropy exhibit upper or lower semicontinuity within the class of mixing interval maps of a fixed modality?

Question 7. Give lower (upper) bounds on the fair entropy of an interval map when all but finitely many points have at least (at most) m preimages. Or can we give some other bounds for this entropy as we did for topological entropy in [C]?

In [MR18] there was always exactly one fair measure (for continuous piecewise monotone and mixing interval maps). In a more general setting, we may have to give up existence or uniqueness of fair measures. But it is possible that for continuous interval maps we might not have to give up existence; we begin by striking out the mixing property:

Question 8. Does every piecewise monotone interval map have a fair measure, even in the non-mixing case?

6 Publications

[A] Jana Doleželová-Hantáková, Zuzana Roth, and Samuel Roth. On the weakest version of distributional chaos. *International Journal of Bifurcation and Chaos*, 26(14):1650235, 2016.

[B] Zuzana Roth. Distributional chaos and dendrites. *International Journal of Bifurcation and Chaos*, 28(14):1850178, 2018.

[C] Samuel Roth and Zuzana Roth. Inequalities for entropy, Hausdorff dimension, and Lipschitz constants. *Studia Mathematica*, 12 pages. To appear in fall 2019.

[D] Ana Rodrigues, Samuel Roth, and Zuzana Roth. Fair measures for countable-to-one maps, 2018. https://arxiv.org/abs/1810.06924.

7 Citations by other authors

1.) Jana Hantáková, Iteration problem for distributional chaos. In International Journal of Bifurcation and Chaos Vol. 27, 2017, ISSN 1750183. (ref. [A], citation by coauthor)

2.) Nilson C. Bernardes Jr., Antonio L. Bonilla, Alfred Peris, Xinxing Wu, Distributional chaos for operators on Banach spaces. In Journal of Mathematical Analysis and Applications, Vol. 459, 2018, ISSN 0022-247X. (ref. [A])

3.) Jianjun Wang, On the Iteration Invariance of Distributional Chaos of Type $2\frac{1}{2}$ in Non-autonomous Discrete System. In Qualitative Theory of Dynamical Systems, Nov 2018, Online ISSN 1662-3592, Print ISSN 1575-5460 (ref. [A])

8 Conferences and other presentations

1.) Real Analysis Exchange, Sarajevo, Bosnia and Herzegovina, June 19 - 25, 2016.
Title of the talk: *On the Weakest Version of Distributional Chaos*

2.) Czech-Slovak Workshop on Discrete Dynamical Systems, Karlova Studánka, Czech Republic, September 12 - 16, 2016.
Title of the talk: *On the Weakest Version of Distributional Chaos*

3.) Indiana University – Purdue University Indianopolis (academic visit) & Midwest Dynamical Systems Conference, Indianapolis, USA, November 1 - 6, 2016.
Title of the talk (Dyn. Sys. Sem. at the university): *Problematic DC-chaos*

4.) 7th Visegrad Conference on Dynamical Systems, Opava, Czech Republic, June 26 - 30, 2017.
Title of the talk: *Li-Yorke sensitivity and a conjecture of Akin and Kolyada*

5.) Dynamical Systems Seminar of Imperial College London (academic visit), London, UK, November 13, 2017.
Title of the talk: *Li-Yorke sensitivity and a conjecture of Akin and Kolyada the results of last year*

6.) University of Exeter (academic visit), Exeter, UK, November 14 - 18, 2017.
Title of the talk (Seminar day in Dynamical Systems): *Distributional Chaos and its weakest version*

7.) Mathematical Association of America Conference (MAATx 2018), Dallas, TX, USA, April 5 - 7, 2018.
Title of the talk: *Distributional chaos and dendrites*

8.) Czech-Slovak Workshop on Dynamical Systems 2018, Banská Bystrica, Slovakia, June 18 - 22, 2018.
Title of the talk: *Distributional chaos and dendrites*

9.) University of Porto (academic visit), Porto, Portugal, Oct. 26 - Nov. 2, 2018.
Title of the talk (Dyn. Sys. Sem. at the university): *Fair Measures for Countable-to-one Maps*

10.) University of Matej Bel, Banská Bystrica, Slovakia, November 11 - 16, 2018.
Title of the talk (Dyn. Sys. Sem. at the university): *Fair Measures for Countable-to-one Maps*

11.) Spring Topology and Dynamical Systems Conference, Birmingham, AL, USA, March 14 - 16, 2019.
Title of the talk: *Distributional chaos and dendrites*

9 References

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